SOLUTION OF EQUATIONS

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PREFACE

The solution of any problem in mathematics is accomplished by an application of one or more fundamental rules which govern the operations involved. These rules may be employed according to a definite plan, in which case the procedure is called a method of solution.

Many problems encountered in industry require a knowledge of not one, but several of the various mathematical subjects. For this reason, this text has been written to include within a single volume all of the fundamental principles of algebra, geometry, trigonometry, logarithms, and analytical geometry of straight lines.

These enumerated sections are not to be thought of as distinct non-related subjects as they are very interdependent. This fact is apparent by noting the frequent use of cross-references between the sections, without which considerable reptition would be necessary.

In practical work the labor of arithmetical computations is greatly reduced by the use of a slide rule. An appendix has been included for explaining the use of this almost indispensable tool, and it is suggested that computations be made with this aid wherever possible.

Tables of the natural trigonometric functions and the logarithms of numbers are included so as to be immediately available for reference.

In the detail preparation of this text special acknowledgment is due to William Coleal, Walter F. McGinty, Ernest J. Gentle and Morrison Perrigo for their diligent and continued efforts towards the completion of this text.

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Section I

ALGEBRA

INTRODUCTION

Algebra is simply the science of calculation by symbols and abbreviations. This branch of mathematics differs from arithmetic particularly in three ways, namely:

In algebra negative as well as positive numbers are used.

In algebra letters of the alphabet are used to represent unknown quantities and in some instances known values as well. Each letter represents a complete number regardless of the number of digits. Two or more letters appearing consecutively indicate the product of those numbers. Thus, ab means (a) multiplied by (b).

In algebra equations are used to express the relationship between two or more quantities.

Like any form of mathematics, algebra consists of four fundamental operations—addition, subtraction, multiplication, and division. These processes are listed in the order in which they are usually found to be most easily performed. This fact should be kept in mind where a choice of solutions is possible.

SYMBOLS AND ABBREVIATIONS

The following symbols and abbreviations are used to indicate the operation to be performed:

Symbol	*Read*	*Operation*
$+$	Plus	Addition
$+$	Positive	
Σ	Summation	Addition
$-$	Minus	Subtraction
$-$	Negative	
$\pm$	Plus or minus	
$\times$	Times or multiplied by	Multiplication
$(\ )$ or $[\ ]$ or $\{\ \}$	The quantity	Multiplication
$\cdot$	Multiplied by	Multiplication

In choosing one of the three symbols to indicate multiplication, the use of parenthesis in most cases is considered best because the letter x is often erroneously interpreted as an unknown term, and the period, as a decimal point.

$\div$ or $/$	Divided by	Division
$\frac{6}{2}$ or $^6/_2$	6 divided by 2	Division
$=$	Equals	
$\sim$ or $\approx$	Approximately equals	
$\neq$	Does not equal	
$>$	Greater than	

$<$	Less than	
$\propto$	Similar to, proportional to, varies as	
∞	Infinity	
$\perp$	Perpendicular to	
$\parallel$	Parallel to	
$\ldots$	And so on	
$\therefore$	Therefore	
$\because$	Since	
$\bigcirc$ $\odot$	Circle, circles	
$\angle$ $\measuredangle$	Angle, angles	
$\llcorner$	Right angle	
$\triangle$ $\triangle$	Triangle, triangles	
Δ	Delta	Difference, or increment
$^{\circ}$	Degrees	
$'$	Minutes or feet	
$''$	Seconds or inches	

In the following symbols and abbreviation the letter (n) is used to represent any unknown quantity or any numerical value

n_x	n sub x	No operation indicated
n_3	n sub 3	Used as shown, (x) and (3) are subscripts to distinguish one (n) from another, each being a different quantity.
n'	n prime	No operation indicated.
n''	n double prime	The prime marks are used to distinguish one (n) from another, each being a different quantity
$5n$	5 times n or $5n$	Multiplication Used as shown, 5 is termed the coefficient of (n).
$1/n$	Reciprocal of n	1 divided by (n).
$\sqrt{}$	Radical sign	Extraction of root.
$\sqrt[r]{n}$	The (r) root of n	Extraction of (r) root of (n) Used as shown, (r) is termed the index of the radical.
$\sqrt{n}$ or $n^{1/2}$	Square root of n	Extraction of square or 2nd root of n. The index 2 is customarily omitted.
$\sqrt[3]{n}$ or $n^{1/3}$	Cube root of n	Extraction of cube or 3rd root.

$\sqrt[4]{n}$ or $n^{1/4}$	Fourth root of n	Extraction of 4th root.
n^2	n squared	Raising (n) to 2nd power or $(n)\ (n)$. Any number or letter occupying the position of 2 as shown is termed the exponent of (n).
n^3	n cubed	Raising n to 3rd power, or $(n)\ (n)\ (n)$.
$\sqrt[3]{n^2}$ or $n^{2/3}$	Cube root of n^2 or square of $\sqrt[3]{n}$ or n to the 2/3 power	Raising n to a fractional power, in this case, the 2/3 power.

Problems involving complex exponents are most easily solved by avoiding the use of the radical sign. Writing all terms in exponential form expedites the solution.

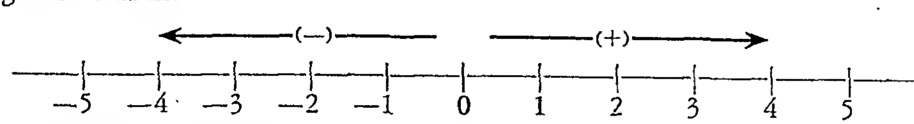
GREEK ALPHABET

Α	α	Alpha	Ι	ι	Iota	Ρ	ρ	Rho
Β	β	Beta	Κ	κ	Kappa	Σ	σ ς	Sigma
Γ	γ	Gamma	Λ	λ	Lambda	Τ	τ	Tau
Δ	δ	Delta	Μ	μ	Mu	Υ	υ	Upsilon
Ε	ε	Epsilon	Ν	ν	Nu	Φ	φ	Phi
Ζ	ζ	Zeta	Ξ	ξ	Xi	Χ	χ	Chi
Η	η	Eta	Ο	ο	Omicron	Ψ	ψ	Psi
Θ	ϑ θ	Theta	Π	π	Pi	Ω	ω	Omega

NUMBERS

Kinds of Numbers

It has already been stated that in algebra negative as well as positive numbers are used, and, like any form of mathematics, algebra involves four fundamental operations —addition, subtraction, multiplication, and division. The rules which govern these operations when such operations are performed with positive numbers are well understood by everyone, but there is a widespread misunderstanding in regard to the use of negative numbers.



To be able to define negative and positive numbers, and to state the rules for operations with such terms it is necessary to introduce the term *absolute value* of a number. Absolute value means the numerical value of a number without regard to its algebraic sign (+) or (—). For example, +5, which is commonly written simply as 5, and —5, have exactly the same absolute values. *Positive* numbers are therefore absolute

values recorded positively from zero, the larger the value, the further the number is above zero *Negative* numbers then are absolute values recorded negatively from zero; the larger the value, the further the number is below zero When the + and − sign of a quantity is taken into account, the term *algebraic value* is employed.

Numbers are also classified as odd or even. This term applies to the absolute value of the number only. Any number is said to be *even* if the number is divisible by 2. Consequently all other numbers are odd.

The number of figures making up a number may also be referred to as the number of *digits*, each figure of the sequence being a digit regardless of the position of the decimal point.

Such numbers as 2, 3, 12, etc. employed in the examples below are termed integers. *Integers* are defined as absolute numerical values that are complete in themselves, being whole numbers without any additional fractional parts Obviously every whole number is an integer, or an integral number.

Two other types of numbers frequently encountered are given the descriptive titles, rational and irrational. A *rational* number is a number that *can* be represented *exactly* by some combination of numbers, either by an integer, a fraction (defined below) or a combination of an integer and a fraction. Thus, 30, ⅞, 1¼, 1250, and 5 50 are all rational numbers.

An irrational number is a number that can not be represented as the quotient of two integers Such numbers can not be expressed exactly by any combination of numbers, either by an integer, a fraction, or a combination of an integer and a fraction. The value of the mathematical symbol π, which is equal to the circumference of a circle divided by its diameter, is an irrational number, for no matter how far the computation is carried out, its value cannot be determined exactly. For practical purposes π is assumed to be 3 1416. Examples of other irrational numbers are $\sqrt{2}$, $\sqrt{3}$, etc.

Fundamental Operations With Numbers

The following rules apply to the use of positive and negative numbers. For convenience, small whole numbers are used for the purpose of explanation, but the rules apply to all quantities regardless of magnitude or whether they are numerical or algebraic terms

Addition: The *sum* of two or more numbers is the result obtained when such numbers are added together Numbers that are added together are called *terms*. The sum of several terms does not depend on the order in which they are added

(1). To add terms of *like* sign together, find the sum of their absolute values and prefix the proper sign (+) or (−), whichever is common to all terms.

$$2+3+4=+9 \qquad \begin{array}{r} 2 \\ 3 \\ \underline{4} \\ 9 \end{array}$$

$$\begin{array}{r} -2 \\ -3 \\ \underline{-4} \\ -9 \end{array} \qquad -2-3-4=-9$$

(2). To add two terms having unlike signs, subtract the term having the smaller absolute value from the term having the larger absolute value, and prefix the sign of the larger. The operation of subtraction is performed according to Rule (3).

$$-5+3=-2 \qquad \begin{array}{r} -5 \\ +3 \\ \hline -2 \end{array}$$

The successive application of the above two rules provides a means to add together a series of terms regardless of sign or number.

Subtraction: The *difference* between two numbers is the result obtained when one number is subtracted from the other. Subtraction is the reverse process of addition, and the numbers are likewise called *terms.* The term to be subtracted is called the *subtrahend,* and the term from which the subtrahend is subtracted is called the *minuend.*

(3). To subtract one term from another, change the sign of the term to be subtracted, and add one term to the other according to Rules (1) and (2) above. This rule is seldom applied to positive numbers as subtracting such terms is understood by everyone. But when one or both of the terms are negative, the rule is indispensable.

$$\begin{array}{r} 5 \\ 2 \\ \hline 3 \end{array} \quad \text{or,} \quad 5-2=3$$

$$\begin{array}{r} -7 \\ -2 \\ + \\ \hline -5 \end{array} \quad \text{or,} \quad -7-(-2)=-7-1(-2)=-7+2=-5$$

Rule (5)

When solving the above example as written in equational form, the term 1 has been inserted in front of the parenthesis signs inclosing the -2. This may clarify the fact that $-(-2)$ is actually equal to $+2$ since $(-1)(-2)=+2$. Rule (5). That 1 may be so inserted (actually or mentally) is apparent if it is considered that (-2) is to be taken 1 times. Also see example, Rule (26), and statement, Page 32.

$$(8a-5b+2c)-(4a-b+4c)=$$

$$\begin{array}{rrr} 8a & -5b & +2c \\ 4a & -\ \ b & +4c \\ - & + & - \\ \hline 4a & -4b & -2c \end{array}$$

The changed signs indicated in the above examples are for purposes of demonstration only. In practice this operation is performed mentally.

Multiplication: The *product* of two or more numbers is the result obtained when these numbers are multiplied together. Numbers that are multiplied together are called *factors.* The product of two or more factors does not depend on the order in

which they are multiplied together. The product of zero and any number of factors is always zero The factor which is to be multiplied is called the *multiplicand*, and the factor by which it is multiplied, the *multiplier*.

(4) The product of *any* number of positive numbers is positive in sign. The absolute value of the product is the product of the several factors.

$$(2)\,(2) = +4$$
$$(2)\,(2)\,(2) = +8$$
$$(2)\,(n) = +2n \qquad \text{Where } n \text{ is any number of positive } (+) \text{ numbers.}$$

(5) The product of any *even* number of negative (—) numbers, by themselves or together with positive numbers, is *positive* in sign.

$$(-2)\,(-2) = +4 \qquad \text{(by themselves)}$$
$$(+2)\,(-2)\,(-2) = +8 \qquad \text{(with positive numbers)}$$

(6) The product of any *odd* number of negative (—) numbers, by themselves or together with positive numbers, is *negative* in sign.

$$(-2)\,(-2)\,(-2) = -8 \qquad \text{(by themselves)}$$
$$(-2)\,(+2) = -4 \qquad \text{(with positive numbers)}$$

Division: The *quotient* of two numbers is the result obtained when one number is divided by the other The number being *divided* is termed the *dividend* The number being *divided by* is termed the *divisor*. Division is the reverse process of multiplication, and the results obtained by division may be checked for accuracy, both as to sign and absolute value, by multiplying the quotient by the divisor according to Rules (4), (5), or (6) to see if it produces the original number.

(7) The quotient obtained when one number is divided by another number of *like* sign is *positive*.

$$\frac{6}{3} = 2$$

$$\frac{-6}{-3} = +2$$

(8) The quotient obtained when one number is divided by another number of *unlike* sign is *negative*

$$\frac{-6}{3} = -2$$

$$\frac{6}{-3} = -2$$

Rules (1) to (8) inclusive apply equally well to numerical and algebraic expressions; the numerical examples being used to clarify the rules. Furthermore, the expressions may be either integers or fractions.

Common Fractions

A fraction may be defined as the indicated quotient of two expressions such as x/y, $4/2$, $-3/5$, $\frac{a}{b-c}$. Thus, x/y means x is to be divided by y. When written in fractional form, either as x/y or $\frac{x}{y}$, the dividend (x) is termed the *numerator,* and the divisor (y) is termed the *denominator.* More simply stated, the expression above the dividing line is the numerator, and that below is the denominator. The numerator and denominator are called the terms of the fraction.

A fraction $\frac{A}{B}$ is called a *rational* fraction when both A and B are rational, a *simple* fraction when A and B are integral, and a *complex* fraction if A or B is fractional. A simple fraction, the numerator of which is of lower degree than the denominator, is called a *proper* fraction. If the degree of the numerator is equal to or greater than that of the denominator, the fraction is an *improper* fraction. An improper fraction can be reduced to an integral expression and a simple fraction.

An indicated division is often called a fraction even though the division can be performed exactly, that is without any remainder, such as $6/3=2$. Any integer can be made a fraction by assuming 1 as a denominator. Thus, $5=5/1$. The result obtained when the answer is not integral may be expressed as a simple fraction, or as an integer together with a simple fraction. Such numbers, consisting of a whole number and a fraction, are called mixed numbers. An example is:

$$x=\frac{25}{4} \text{ or } 6\tfrac{1}{4}$$

Fractions composed of letters are usually spoken of as algebraic fractions and those composed of definite numerical values are called numerical fractions. However, there is no difference in the manner in which they are treated in order to solve the problems in which they occur. In fact, it is a recommended practice, when in doubt about an operation in algebraic fractions, to perform a similar operation using numerical values which will allow a rapid check of the method and from this determine what the operation with the algebraic expression should be.

The operations described in the following pages are valid for both simple and complex fractions, algebraic or numerical. The examples employing algebraic fractions are paralleled by similar examples employing numerical fractions. This plan has been followed because the results of the operations with the numerical values are obvious. These results then serve to substantiate the rules as given.

(9). The value of a fraction is not altered by multiplying *both* the numerator and the denominator by the same number or by the same (or equal) expression.

$$\frac{x}{y}=\frac{3x}{3y}$$

Since:

$$\frac{4}{2}=\frac{(3)\,(4)}{(3)\,(2)}=\frac{12}{6}=2$$

$$\frac{5}{\sqrt{2}}=\frac{5\sqrt{2}}{\sqrt{2}\sqrt{2}}=\frac{5\,(1.414)}{2}=3.535$$

(Note: $\sqrt{2}=1.414$ approximately)

(9a) The value of a fraction is not altered by changing the sign of both the numerator and the denominator. This operation is equivalent to multiplying both the numerator and the denominator by the quantity (-1).

$$\frac{6}{2}=\frac{-6}{-2}=3$$

(9b) The value of a fraction is not altered by changing the sign before the fraction and also the sign of either the numerator or the denominator.

$$\frac{6}{2}=-\frac{-6}{2}=-\frac{6}{-2}=3$$

(10) The value of a fraction is not altered by dividing *both* the numerator and the denominator by the same number or by the same (or equal) expression.

$$\frac{x}{y}=\frac{x/3}{y/3}$$

Since
$$\frac{12}{6}=\frac{12/3}{6/3}=\frac{4}{2}=2$$

$$\frac{3\sqrt{3}}{5\sqrt{4}}=\frac{3\sqrt{4}\sqrt{3}}{5\sqrt{4}\sqrt{4}}=\frac{(3)\,(2)\sqrt{3}}{(5)\,(4)}=\frac{6\sqrt{3}}{20}=\frac{3\sqrt{3}}{10}=$$

$$\frac{(3)\,(1.732)}{10}=\frac{5.196}{10}=.5196$$

(Note $\sqrt{3}=1.732$ approximately)

In dealing with common fractions, it is important to know that there is *no* rule which states that both the numerator and denominator of a fraction may be raised to any power, or that any root may be taken of both numerator and denominator without altering the value of the fraction. Such operations are frequently erroneously performed. The fallacy of such an operation is shown by an example on Page 30.

In order to solve certain types of expressions and equations involving fractions, it is necessary that all terms are written so as to have a *common denominator.* Such a denominator must be identical for each term, and of such magnitude as to include each denominator of the original terms an exact number of times. Thus 12 is a common denominator for 2/3 and 3/4.

It is apparent that a common denominator for two or more fractions can always be obtained by multiplying together each denominator of the several fractions, since such a product would obviously include each denominator of the original terms an exact number of times. A common denominator can also be obtained by multiplying together each *different* denominator of the several fractions. These two methods are often employed in adding or subtracting a series of fractions having dissimilar denomi-

nators. The results obtained by these methods are correct but are not always expressed in their simplest form and therefore require additional operations. To eliminate this deficiency, the denominator employed should be chosen so that it exactly contains each denominator of the given terms, yet is no larger than necessary to do so. Such a denominator is called the *least common denominator* of the several fractions, or abbreviated, L. C. D.

The method of determining the L. C. D. of two or more fractions employs the use of the term factors and prime factors which are now explained and defined:

The term *factor* is defined as a quantity which can be exactly divided into the given term. Thus 3 is a factor of 12, since $12/3 = 4$. Factors, for some purposes, are not always integers, but must be considered as integral numbers at this time. The quotient (4) obtained in this case can be further factored into (2) (2), but can be factored no further.

When any quantity is completely factored, the *prime* factors are obtained. Such factors are easily recognized since they are exactly divisible only by themselves or by one.

Example:

$$60 = (5)\ (3)\ (2)\ (2)$$

(11). The value of the least common denominator for any number of fractions is readily found as follows:

Step 1. Find the prime factors of each of the denominators when each fraction is represented in its simplest form.

Step 2. Find the product of all the different prime factors, using each factor the greatest number of times it occurs in any one denominator.

$$\frac{2}{3} + \frac{3}{4} - \frac{4}{9} = \frac{2}{3} + \frac{3}{2 \cdot 2} - \frac{4}{3 \cdot 3}$$

$$\text{L. C. D.} = (2)\ (2)\ (3)\ (3) = 36$$

$$= \frac{24}{36} + \frac{27}{36} - \frac{16}{36} \qquad \text{Rule (9)}$$

The failure to represent each fraction in its simplest form will result in finding a number which is not the L. C. D. and the full advantage of the method will not be realized.

(12). The sum of two or more fractions having a common denominator is a fraction whose numerator is the sum of the numerators of the fractions to be added, and whose denominator is the common denominator.

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

Since:
$$\frac{6}{2} + \frac{4}{2} = \frac{6+4}{2} = \frac{10}{2} = 5$$

(13). The difference between two fractions having a common denominator is a fraction whose numerator is the difference between the numerators of

the fractions to be subtracted, and whose denominator is the common denominator.

$$\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$$

Since $$\frac{6}{2}-\frac{4}{2}=\frac{6-4}{2}=\frac{2}{2}=1$$

The sum or the difference of any two or more fractions having dissimilar denominators is usually found by first reducing the given fractions to equivalent fractions all having a common denominator. The following rules, (14) and (15), however, can be used to advantage when only two such fractions are involved

(14). The *sum* of any two fractions having dissimilar denominators, such as a/b and c/d is $\frac{ad+bc}{bd}$. This is shown to be true by the application of Rule (9) and Rule (12).

$$\frac{a}{b}=\frac{ad}{bd} \text{ and } \frac{c}{d}=\frac{bc}{bd} \qquad \text{Rule (9)}$$

$$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd} \qquad \text{Rule (12)}$$

$$\frac{3}{4}+\frac{2}{3}=\frac{9}{12}+\frac{8}{12}=\frac{17}{12}=1\frac{5}{12}$$

By rule: $a=3, b=4, c=2, d=3$

$$\frac{3}{4}+\frac{2}{3}=\frac{(3)\,(3)+(4)\,(2)}{(4)\,(3)}=\frac{9+8}{12}=\frac{17}{12}=1\frac{5}{12}$$

(15) The *difference* between any two fractions having dissimilar denominators, such as a/b and c/d is $\frac{ad-bc}{bd}$.

$$\frac{a}{b}-\frac{c}{d}=\frac{ad}{bd}-\frac{bc}{bd}=\frac{ad-bc}{bd} \qquad \text{Rule (14)}$$

$$\frac{3}{4}-\frac{2}{3}=\frac{9}{12}-\frac{8}{12}=\frac{1}{12}$$

By rule $a=3, b=4, c=2, d=3$.

$$\frac{3}{4}-\frac{2}{3}=\frac{(3)\,(3)-(4)\,(2)}{(4)\,(3)}=\frac{9-8}{12}=\frac{1}{12}$$

(16). The *product* of two or more fractions is another fraction whose numerator is the product of the separate numerators and whose denominator is the product of the denominators

$$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right)=\left(\frac{ac}{bd}\right)$$

Since: $$\left(\frac{4}{2}\right)\left(\frac{6}{3}\right)=\frac{(4)\ (6)}{(2)\ (3)}=\frac{24}{6}=4$$

Check $$(2)\ (2)=4$$

(16a). The product of a fraction and an integer is a fraction whose numerator is the product of the numerator of the given fraction and the integer, and whose denominator is the denominator of the given fraction.

$$\left(\frac{3}{4}\right)(5)=\frac{(5)\ (3)}{4}=\frac{15}{4}$$

Since: $$\left(\frac{3}{4}\right)(5)=\left(\frac{3}{4}\right)\left(\frac{5}{1}\right)=\frac{15}{4}$$ Rule (16)

(16b). The product of a fraction and an integer is a fraction whose numerator is the numerator of the given fraction, and whose denominator is the quotient of the denominator of the given fraction and the integer.

$$\left(\frac{7}{8}\right)(4)=\frac{7}{8/4}=\frac{7}{2}=3.5$$

Since: $$\left(\frac{7}{8}\right)\left(\frac{4}{1}\right)=\frac{28}{8}=\frac{7}{2}=3.5$$ Rule (16)
Rule (16a)

(17). The *quotient,* when one fraction is divided by another, is obtained by inverting the fraction which is the divisor and then multiplying according to Rule (16).

$$\frac{a}{b}\div\frac{c}{d}=\left(\frac{a}{b}\right)\left(\frac{d}{c}\right)=\frac{ad}{bc}$$

Since: $$\frac{6}{3}\div\frac{2}{1}=\left(\frac{6}{3}\right)\left(\frac{1}{2}\right)=\frac{6}{6}=1$$

Check $$2\div 2=1$$

(17a). The quotient, when a fraction is divided by an integer, is a fraction whose numerator is the numerator of the given fraction divided by the integer, and whose denominator is the denominator of the given fraction.

$$\frac{8}{15}\div 4=\frac{8/4}{15}=\frac{2}{15}$$

Since: $$\frac{8}{15}\div 4=\frac{8}{15}\div\frac{4}{1}=\left(\frac{8}{15}\right)\left(\frac{1}{4}\right)=\frac{8}{60}=\frac{2}{15}$$ Rule (17)

(17b). The quotient, when a fraction is divided by an integer, is a fraction whose numerator is the numerator of the given fraction, and whose denominator is the product of the integer and the denominator of the given fraction.

$$\frac{9}{2}\div 5=\frac{9}{(5)\ (2)}=\frac{9}{10}$$

Since $$\frac{9}{2} \div 5 = \frac{9}{2} \div \frac{5}{1} = \left(\frac{9}{2}\right)\left(\frac{1}{5}\right) = \frac{9}{10}$$

In the proof of Rules (16) and (17) above, use is made of the fact that an integer can always be written as a fraction by simply assuming unity (1) as a denominator. This is obviously a valid operation and can be used whenever desired.

(18) The *reciprocal* of a fraction is merely the fraction inverted.

$$\frac{1}{a/b} = \left(\frac{1}{1}\right)\left(\frac{b}{a}\right) = \frac{b}{a} \qquad \text{Rule (17)}$$

Since: $$\frac{1}{2/4} = \left(\frac{1}{1}\right)\left(\frac{4}{2}\right) = \frac{4}{2} = 2$$

Check $$\frac{1}{1/2} = 2$$

Decimal Fractions

The fundamental operations of addition, subtraction, multiplication, and division of numbers involving fractions can be greatly simplified if these fractions are written as decimals Furthermore, only decimal fractions can be used for operations performed with a slide rule

A decimal fraction is similar to a common fraction except that in a decimal fraction the denominator is always 10, 100, 1000, etc. In writing a decimal fraction it is convenient to omit the denominator and indicate its value by placing a point (.), called a decimal point, in the numerator so that there are as many figures to the right of this point as there are zeros in the denominator

Thus, $\frac{5}{10}$ is written .5 or 0.5, $\frac{25}{100} = .25$ or 0.25,

$$\frac{75}{1000} = .075 \text{ or } 0.075, \text{etc}$$

The zero, which is written to the left of the decimal point in many cases for clearness, is not necessary and may be omitted.

The number of figures, including zeros, to the right of the decimal point are called decimal places It should be noted that when there are fewer figures in the numerator than there are zeros in the denominator, zeros are inserted to the right hand side of the decimal point, between the decimal point and the figures making up the decimal fraction, to make the required number of decimal places. For example:

$$\frac{75}{1000} = .075$$

It is apparent that the position of the decimal point is the factor controlling the numerical value of any given sequence of figures For each place the point is moved to the right, the value of the decimal fraction is multiplied by 10; and for each place it is moved to the left, the value is divided by 10

Thus, 2.5 becomes 25 when the decimal point is moved one place to the right, and 0.25 when the point is moved one place to the left In the first case, 2.5 is multiplied by 10, and in the second case, it is divided by 10.

Multiplication and division by 100 and 1000, etc., is similarly performed. For every place the decimal point is moved toward the right, the value of the given number is increased ten times, and for every place the decimal point is moved to the left, the value of the given number is decreased ten times. Therefore, the relative values of the various places to the right and left of the decimal point are as follows:

Thousands	Hundreds	Tens	Units	Decimal point	Tenths	Hundredths	Thousandths	Ten-thousandths	Hundred-thousandths	Millionths	Ten-millionths	Hundred-millionths
0	0	0	0	.	0	0	0	0	0	0	0	0

A decimal fraction, or a mixed number consisting of a whole number and a decimal fraction, can always be obtained from a common fraction by dividing the numerator by the denominator. Zeros may be added to the numerator and the division continued until the desired degree of accuracy is attained.

The operation of changing a decimal fraction into a common fraction of the same value is not frequently required, but where necessary can be obtained by one of the following two rules.

(19). To change a decimal into a common fraction, write the given sequence of figures without any decimal point as the numerator. The denominator is the integer 1 with as many zeros annexed at its right as there are decimal places in the given decimal fraction. In many cases, the resulting fraction can be simplified by dividing both the numerator and denominator by the same number.

$$.25 = \frac{25}{100} = \frac{1}{4}$$

(20). To change a decimal into a common fraction having a specified denominator, multiply the decimal by a common fraction which has both a numerator and denominator equal to the specified denominator. The product is the equivalent common fraction.

$$.8125 \times \frac{16}{16} = \frac{13}{16}$$

Operations involving decimal fractions are performed according to the following rules:

(21). To add decimals, place the numbers to be added in a column so that their decimal points are in line under one another. Then add as if they were whole numbers, and place the decimal point in the sum directly under the decimal points of the numbers being added.

.375	15.32	50 057
.253	2.756	0 023
.628	18 076	50.080

(22) To subtract one decimal from another, place the number to be subtracted below the number being subtracted from so that their decimal points are in line. Then subtract as if they were whole numbers, and place the decimal point in the difference directly under the decimal points of the numbers above

.375	15 32	50 057
253	2.756	0 023
122	12 564	50 034

(23) To multiply decimals, find their product as if they were whole numbers, and then point off as many decimal places in the product as the sum of the decimal places in the factors being multiplied together. Zeros may be inserted between the decimal point and the figures obtained as a product to make the required number of decimal places

.2	25	12 5
3	03	.5
06	.0075	6 25

(24) To divide one decimal by another, find the quotient as if they were whole numbers Zeros may be added to the dividend and the division continued until the desired degree of accuracy is attained. The position of the decimal point in the quotient can usually be found by inspection. The assumed position of the decimal point can be checked for validity by multiplying the quotient by the divisor to obtain the original dividend In division of decimals, it is often convenient to change the divisor to a whole number by moving the decimal point to the right as many places as there are figures in the decimal The decimal point in the dividend must be moved an equal number of places to the right, counting from the original position If there are fewer figures to the right of the decimal point in the dividend than there are to the right of the decimal point in the divisor, annex enough zeros to the right of the dividend to take care of the new decimal point. Then divide as in whole numbers

$$25\overline{)\,75}$$

$$\begin{array}{r} 30 \\ 25\overline{)\,750.} \\ \underline{75} \\ 0 \end{array}$$

Square Root of Numbers

The frequent need for finding the square root of numbers makes it desirable to know a method of solution that does not require the use of tables, slide rule, or calculating machine.

The square roots of the numbers 1, 4, 9, 16, 25, 36, 49, 64, and 81, are 1, 2, 3, 4, 5, 6, 7, 8, and 9 respectively. Such numbers as those mentioned, which have exact square roots, are termed *perfect squares.* The square root of any number which is not a perfect square, such as 37, 11, etc., cannot be *exactly expressed,* although its approximate value can be found to any number of decimal places, and therefore, quite accurately.

The explanation and solution of several problems in square root, given below, indicates the procedure in finding the square root of any number, whether it be a perfect square or not. For imperfect squares, the root should be found to one more decimal place than is desired in the answer, so that the final figure of the root may be increased or left unaltered according to whether the additional figure obtained in the root is greater or less than 5.

Example 1:

$$\sqrt{1190.25}$$

Solution: $\sqrt{11'90.25}$

Step 1.

First separate (by actual indications as shown, or mentally) the number into *periods* of two figures each, commencing at the decimal point and going both to the right and to the left. The extreme left period may consist of two digits, as in this example, or only one digit as in Example 2. The extreme right period should consist of two digits, as a zero can always be annexed where an additional figure is necessary to complete the period.

Step 2.

$$\begin{array}{l} \;\;3 \\ \sqrt{11'90.25} \\ \;\;\underline{9} \\ \;\;2\,90 \end{array}$$

Find the largest perfect square which is not greater than the first period on the left (the largest perfect square contained in 11 is 9) and write its *square root* ($\sqrt{9} = 3$) *above* the first period, this number being the first figure of the required root. The square of this number ($3^2 = 9$) is placed *under* the first period and subtracted from it ($11 - 9 = 2$). Now bring down the second period (90) and annex, (not add), it to the remainder (2), thus obtaining (290), termed the *first remainder.*

Step 3.

$$\begin{array}{ll} & \;\;3\;\;4 \\ & \sqrt{11'90.25} \\ & \;\;\underline{9} \\ 64 & \;\;2\,90 \\ & \;\;\underline{2\,56\;\;\;\;} \\ & \;\;\;\;34\,25 \end{array}$$

Take twice the part of the root already found ($2 \times 3 = 6$) and use it for a *trial divisor,* writing it at the left of the first remainder. Find how

many times this trial divisor (6) is contained in the first remainder (290) without its right-hand figure (0) Obviously, 6 is contained in 29 four times. This number (4) is placed *above* the second period and becomes the second figure of the required root. This same number (4) is also placed to the right of the trial divisor, making the *true divisor* 64. Now multiply this true divisor (64) by the last figure placed in the root (4), and write the product ($64 \times 4 = 256$) *under* the first remainder from which it is subtracted, the difference being 34. As before, bring down and annex the next period (25) to this remainder, producing a *second remainder* of 3425.

Step 4

$$
\begin{array}{lr}
 & 3\ \ 4\ \ 5 \\
 & \sqrt{11'90'25} \\
 & 9 \\
\cline{2-2}
64 & 2\,90 \\
 & 2\,56 \\
\cline{2-2}
685 & 3425 \\
 & 3425 \\
\cline{2-2}
\end{array}
$$

As in Step 3, take twice the part of the root so far found ($2 \times 34 = 68$) and use it for a second trial divisor, writing it at the left of the second remainder (3425). Find how many times this trial divisor (68) is contained in the second remainder (3425) without its right-hand figure (5) 68 is contained in 342 five times This number (5) is placed above the third period and becomes the third figure of the required root This same number (5) is also placed to the right of the trial divisor, making the true divisor 685 Now multiply this true divisor (685) by the last figure placed in the root (5), and write the product ($685 \times 5 = 3425$) under the second remainder from which it is subtracted, the difference being zero Since there is no remainder, and there are no more periods in the given number (1190.25) to be brought down, the solution is complete It is apparent the number 1190 25 is a perfect square inasmuch as it has a perfect square root

Example 2

$$\sqrt{589.321}$$

Solution $\sqrt{589.321} = \sqrt{589\,321000}$

$$\sqrt{5'89\,32'10'00}$$

Step 1.

First separate (by actual indication as shown, or mentally) the number into *periods* of two figures each, commencing at the decimal point and going both to the right and to the left The extreme left period, as in this example, may consist of only one digit, but is nevertheless treated as a regular two digit period; the extreme right period should consist of two digits, as a zero can always be annexed where an additional figure is necessary to complete the period.

The *number* of decimal places in the root of a number which is not a perfect square is controlled by the number of periods of zeros annexed at the right of the number. The decimal point in the root will be *placed* so that there are as many digits to the left of the decimal point as there are whole number periods in the number of which the root is desired. In the example, the decimal point in the root will be located so as to provide two figures to the left of the decimal point, since the given number contains two whole number periods.

Step 2.

$$\begin{array}{l} \quad 2 \\ \sqrt{5'89.32'10'00} \\ \quad \underline{4} \\ \quad 1\,89 \end{array}$$

Find the largest perfect square which is not greater than the first period on the left (largest perfect square contained in 5 is 4) and write its *square root* ($\sqrt{4} = 2$) *above* the first period, this number being the first figure of the required root. The square of this number ($2^2 = 4$) is placed *under* the first period and subtracted from it ($5 - 4 = 1$). Now bring down the second period (89) and annex (not add) it to the remainder (1), thus obtaining (189), termed *first remainder.*

Step 3.

$$\begin{array}{ll} & \quad 2\;\;4 \\ & \sqrt{5'89.32'10'00} \\ & \quad \underline{4} \\ 44 & \quad 1\,89 \\ & \quad \underline{1\,76\;\;} \\ & \quad\;\; 13\,32 \end{array}$$

Take twice the part of the root already found ($2 \times 2 = 4$) and use it for a *trial divisor,* writing it at the left of the first remainder. Find how many times this trial divisor (4) is contained in the first remainder (189) without its right-hand figure (9). Obviously, 4 is contained in 18 four times. This number (4) is placed *above* the second period and becomes the second figure of the required root. This same number (4) is also placed to the right of the trial divisor, making the *true divisor* 44. Now multiply this true divisor (44) by the last figure placed in the root (4) and write the product ($44 \times 4 = 176$) *under* the first remainder from which it is subtracted, the difference being 13. As before, bring down and annex the next period (32) to this remainder, producing a *second remainder* of 1332.

Step 4.

$$\begin{array}{ll} & \phantom{\sqrt{}}2\ \ 4\ \ 3 \\ & \sqrt{5\,89\,32\,10\,00} \\ & 4 \\ 44 & 1\,89 \\ & 1\,76 \\ 483 & \ \ 13\,32 \\ & \ \ 14\,49 \end{array}$$

As in Step 3, take twice the part of the root so far found ($2 \times 24 = 48$) and use it for a second trial divisor, writing it at the left of the second remainder (1332). 48 is contained in 133 three times. This number (3) is placed *above* the third period and becomes the third figure of the required root. This same number (3) is also placed to the right of the second trial divisor, making the true divisor 483. Now multiply this true divisor (483) by the last figure placed in the root (3) and write the product ($483 \times 3 = 1449$) under the second remainder. It is obviously larger than the second remainder (1332), and cannot be subtracted from it with a positive result. Three is therefore not the third figure of the root, and this step must be repeated using a smaller number than 3.

(Whenever the product of the trial divisor and the last placed figure in the root exceeds the corresponding remainder, the root number chosen is too great, and a repetition of this step in the process is necessary using a smaller figure.)

Step 5.

Repeat Step 4 using 2 as the third figure of the root

$$\begin{array}{ll} & \phantom{\sqrt{}}2\ \ 4\ \ 2 \\ & \sqrt{5'89'32'10'00} \\ & 4 \\ 44 & 1\,89 \\ & 1\,76 \\ 482 & \ \ 13\,32 \\ & \ \ \ \ 9\,64 \\ & \ \ \ \ 3\,68\,10 \end{array}$$

Steps 6, 7, ...

Continue the process until all periods of the number have been brought down, (including zero periods annexed to provide the desired number of decimal places in the root).

The complete solution appears as follows

	2 4. 2 7 5
	$\sqrt{589.321000}$
	4
44	189
	176
482	1332
	964
4847	36810
	33929
48545	288100
	242725

Since there will be a remainder when the quantity 242725 is subtracted from 288100, it is not necessary to continue the solution any further as it is obvious that additional numbers to follow 5 can be found. The number 5 therefore, represents some value actually greater than 5. Consequently, if the answer is to be written to two places to the right of the decimal point, it will be expressed as 24.28.

Example 3:

$$\sqrt{39601.}$$

It sometimes appears that a step in the solution for the root is in error, as for example in the finding of the second figure of the square root of 39601.

	1
	$\sqrt{3'96'01}$
	1
2	2 96

The rule as previously stated, is to see how many times twice the first figure of the root is contained in the first remainder without its right-hand figure. In this case it would be 14 as $29/2 = 14+$. This is impossible as there can never be a two-digit number placed in the root as one of the figures, and 9 is therefore the largest number which it is possible to use.

	1 9 9.
	$\sqrt{3'96'01.}$
	1
29	2 96
	2 61
389	35 01
	35 01

Example 4:

$$\sqrt{0.091204}$$

$$\begin{array}{rl} & .\,3 \\ & \sqrt{0\,09'12'04} \\ & 9 \\ 6 & 12 \\ & 12 \end{array}$$

Whenever the trial divisor is not contained in the corresponding remainder without its right-hand figure, place a zero in the root, and also a zero at the right of the trial divisor, then bring down the next period, and continue as before

$$\begin{array}{rl} & .\,3\ \ 0\ \ 2 \\ & \sqrt{0\,09'12'04} \\ & 9 \\ 602 & 12\,04 \\ & 12\,04 \end{array}$$

The finding of the square root of a decimal fraction is sometimes more easily solved if the position of the decimal point is changed so that the number operated on is, at least in part, a whole number. The application of this method is as follows:

$$\sqrt{0\,091204}$$

Rule. To obtain the square root of decimal fractions, first move the decimal point an *even number* of places to the right so that the number is expressed as some whole number between 1 and 100 Find the square root of this number and then move the decimal point *half* as many places to the left as it was moved to the right in the first place This is the square root of the given decimal fraction.

$$\begin{array}{l} 3.\ 0\ 2 \\ \sqrt{009.1204} \end{array}$$

$$\begin{array}{l} .3\,0\,2 \\ \sqrt{0\,091204} \end{array} \qquad \text{Answer}$$

The decimal point could have been moved *any even* number of places to the right to form a whole number, and not necessarily that even number of places to produce a number between 1 and 100 After finding the numerical value of the root of the number so formed, the position of the decimal point is moved half as many places to the left as it was moved to the right in the first place. This will be the square root of the given number. The rule as first stated is recommended in preference to this latter method since slightly less labor is involved.

Example 5:

The square root of a fraction in which either the numerator or the denominator, or both, are perfect squares are solved most easily as shown below:

$$\sqrt{\frac{100}{49}} = \frac{\sqrt{100}}{\sqrt{49}} = \frac{10}{7} = 1.43 \text{ (approx)}$$

This is somewhat less laborious in most cases than if the decimal equivalent of the fraction is found first and the square root then obtained of the resulting decimal.

If only the numerator of the fraction is a perfect square, then the logical procedure is not so evident as in the previous example. However, it is recommended that the square root of the numerator and the denominator be separately found, the square root of the perfect square being found mentally, and the square root of the other term being found by slide rule, logarithms, or by arithmetical calculation depending upon the degree of accuracy required. The value of the resulting fraction is then found by dividing the numerator by the denominator.

After finding the square root (or any *even* root such as $\sqrt[4]{\ }$, $\sqrt[6]{\ }$, etc.) the prefix $\pm$ may be assigned to the absolute value as found since the root may be either $(+)$ or $(-)$. This is true because the product of any number of positive numbers is positive in sign, and the product of any *even* number of negative numbers, by themselves or together with positive numbers, is positive in sign.

A check on the accuracy of the absolute value of the root found from any calculation may be accomplished by squaring the computed value to see if it produces the original number. In the special case where a number ending in five is to be squared, a useful method is available which simplifies the necessary arithmetical work. The steps in the application of this method are:

Step 1. Write the given sequence of digits omitting the 5. Call the resulting number 'n'.

Step 2. Determine the product of $n(n+1)$.

Step 3. Annex 25 to the right of the product obtained in Step 2. The number thus obtained is the square of the given number.

$$\begin{array}{r} 245 \\ \underline{245} \end{array}$$

$$(24)\ (24+1) = (24)\ (25) = 600$$
$$(245)^2 = 60025$$

EXPONENTS

The exponent of a quantity indicates the number of times the quantity is to be multiplied by itself, as:

$$a^3 = (a)\ (a)\ (a)$$

If the exponent has an absolute numerical value greater than one, it is known as a power, and if less than one, the exponent is called a root. However, in many cases the word power is extended to include fractional exponents. (This usage of the term root is not to be confused with the root of an equation defined elsewhere).

Exponents may be integers, fractions, or a combination of the two, or they also may be algebraic expressions including the logarithms of quantities. Regardless of nature, they are placed to the right and above the term which they affect. Where possible, their size should be less than the size of the term itself. The radical sign should also be thought of as an exponent as it has the same effect as though the ½ power were indicated.

Algebraic operations involving terms to which exponents are affixed must be per-

formed in accordance with a definite set of rules if valid results are to be obtained. Because of their great importance in algebra and in logarithms (Section IV) the application of these rules should be thoroughly understood.

(25) The range of influence of the exponent or radical sign extends only over the term to which it is adjacent.

$$2ax^2 = (2a)\,(x^2)$$
$$\sqrt{4}a = (\sqrt{4})\,(a) = \pm 2a \; (\pm) \quad \text{Rule (4) (5)}$$

(26). An expression within parenthesis, brackets, braces, a radical sign, or overscored or underscored with a horizontal line is to be treated as a single quantity. This rule is frequently applied when dealing with exponents, but is equally important in all algebraic operations.

$$(ab)^2 = a^2b^2$$
$$(a+b)^2 = (a+b)\,(a+b)$$
$$\sqrt{4x^2} = \sqrt{4}\,\sqrt{x^2} = \pm 2x \quad (\pm) \quad \text{Rule (4) (5)}$$
$$-(-6+8) = 6-8 = -2$$
$$\sqrt{b^2-4a\,(-c)} = \sqrt{b^2-(4a)\,(-c)} = \sqrt{b^2+4ac} \qquad \text{Rule (5)}$$
$$\left(\frac{18}{R}\right) b = \frac{18b}{R} \qquad \text{Rule (16a)}$$

(27) Positive or negative identical terms with the same exponents are added according to the rules of positive and negative numbers, Rule (1) and Rule (2).

$$2x^2 + x^2 = 3x^2$$

(28) Positive or negative identical terms with the same exponents are subtracted according to the rule of positive and negative numbers, Rule (3).

$$2x^2 - x^2 = x^2$$

(29). The product of two or more like quantities (which have either like or unlike exponents) is equal to this quantity with the sum of the exponents of the factors as an exponent

$$(2)\,(2) = 2^{1+1} = 2^2 = 4$$
$$(-2)\,(-2)\,(-2) = -2^{1+1+1} = -2^3 = -8$$
$$(2x)\,(x^2) = (2)\,(x)\,(x^2) = 2x^{1+2} = 2x^3$$

(30). The product of two or more unlike quantities, each having the same exponent, is equal to the product of these quantities with the same exponent as the exponent of the factors being multiplied together.

$$(2)^3\,(3)^3 = (6)^3 = 216$$
$$(8)\,(27) = 216$$
$$(a)^2\,(ab^3) = a^3b^3 = (ab)^3$$

(31). The quotient of two like quantities (which have either like or unlike exponents) is equal to this quantity with the difference between the exponents of the dividend minus the divisor as an exponent.

$$3^3 \div 3^2 = 3^{3-2} = 3$$
$$\text{Since} \qquad 27 \div 9 = 3$$
$$x^{1.5} \div \sqrt{x} = x^{1.5} \div x^{.5} = x^{1.5\,-.5} = x$$

(32). The quotient of two unlike quantities, each having the same exponent, is equal to the quotient of these quantities with the same exponent as the exponents of the quantities being divided.

$$(4)^2 \div (2)^2 = (4/2)^2 = 4$$
$$\text{Since} \qquad 16 \div 4 = 4$$
$$x^2 \div y^2 = (x/y)^2$$

(33). Rules (31) and (32) may be combined into a single rule which has the advantage of being more understandable and workable in many cases:

When changing any quantity from the numerator to the denominator of a simple fraction, or vice versa, the plus or minus ($\pm$) of the exponent of that term is reversed. The fraction can then be simplified according to the rules applying to multiplication, Rules (29) and (30).

$$\frac{x^3}{x^2} = (x^3)\ (x^{-2}) = x^1$$

$$\frac{b^{-2}}{2a^{-1}} = \frac{a}{2b^2}$$

The above rule can be expanded to include other than simple fractions provided that the numerators and denominators of such terms are products of terms. Such a fraction excludes any terms in addition or subtraction.

$$\frac{x^2y}{2x} = \frac{xy}{2}$$

$$\frac{3x^{-2}}{a^{-1}} = \frac{3a}{x^2}$$

Rule (33) is useful in establishing the numerical value to be assigned to any term raised to the zero power. It is common knowledge that any quantity divided by itself is one.

$$\frac{x}{x} = 1$$

Also by the above rule:

$$\frac{x^1}{x^1} = (x^1)\ (x^{-1}) = x^0$$

Therefore since $x/x = 1$, then x^0 must also $= 1$, since $x/x = x^0$. (Things equal to the same thing, or equal things, must be equal to each other.)

(34) Any quantity raised to any power is equal to the quantity with its original exponent multiplied by the power in question.

$$(2)^2 = (2^1)^2 = 2^{(1)(2)} = 2^2 = 4$$
$$(2^2)^3 = 2^{(2)(3)} = 2^6 = 64$$
$$\text{Since} \quad (4)^3 = 64$$

(35) Any quantity from which a given root is to be extracted is equal to the quantity with its exponent divided by the root in question.

$$\sqrt[3]{2^6} = 2^{6/3} = 2^2 = +4$$
$$\sqrt[3]{64} = +4$$
$$\sqrt{x} = x^{1/2}$$
$$\sqrt[n]{x} = x^{1/n}$$

WRITING OF EQUATIONS USING SYMBOLS AND ABBREVIATIONS

Using the conventional symbols and abbreviations as described, it is possible to reduce to algebraic form any problem in which there exists a mathematical relationship. Such forms may be classed as either expressions or equations. An algebraic *expression* in its simplest form is a statement that some term exists, examples of which are a, x, and 2 In other cases an expression denotes that some operation is to be performed, such as $\sqrt{2}$, $(a - b)$, $(x - 3c)$. An *equation* is defined as a statement of equality between two expressions, that is, that two expressions are equal. $x = y$, $x - y = 25$, $a = o$ An equation can always be distinguished from an expression by the presence of the equal sign together with an accompanying term, even though this term is zero. Without this accompanying term an equation becomes an expression, thus $(x - 7 = \ )$ is an expression no different from $x - 7$

Equations may be either *conditional equations* or *identities*. An equation that is true for only certain values of the unknown involved is termed a conditional equation. Identities are equations which are satisfied by all values assigned to the unknown Identities cannot be solved because, when simplified, they become $0 = 0$. At the opposite extreme from identities stand *descriptions* which are not true equations because they cannot be satisfied by any number. When the word equation is used without further qualification, it is a conditional equation that is implied.

Equations are also classified according to their degree. The degree of an equation containing only one unknown is the same as the numerical value of the largest exponent of that unknown The equation $x = 4$ is an equation of the first degree. Such equations are also termed linear equations since all of its plotted points will fall on a straight line. The equation $ax^2 + bx + c = 0$ is called a quadratic equation or an equation of the second degree. An equation containing x^3 is called a cubic equation or an equation of the third degree

Whenever an unknown quantity is to be represented in either an expression or an equation it is customary to assign one of the last letters of the alphabet, as x, y, or z to indicate this quantity Numerical quantities of known value are sometimes represented by the first letters of the alphabet, a, b, c, . . . , but it is generally advisable to use the known values directly until doing otherwise is justified by experience. The assignment of letters to represent unknown quantities as well as the proper use of known values may be shown by an example:

4 times the square of a certain unknown number minus 9 times that number is equal to -2

Let x represent the unknown number

$$4x^2 - 9x = -2$$

It is evident that the equation expresses by a few symbols all that was formerly conveyed by a great number of words. This form also presents less possibility for misinterpretation of facts. The greatest advantage of the algebraic representation is that it allows a rapid and precise solution of the problem in question.

Rules cannot be definitely set forth in the writing of equations as they can in the methods of solution. To write an equation, the conditions of the problem must be understood, and also, the person writing the equation must know the signs and symbols —the algebraic language. This second requisite will be acquired with the study of the subjects throughout the text.

The following suggestions may be helpful as a guide when attempting to write an algebraic equation representing a mathematical relationship.

Step 1.
Read carefully the statement of the problem as given in words.

Step 2.
Determine what is the unknown quantity or quantities, and represent these by letters of the alphabet, always using the minimum number of letters. This is accomplished by expressing as many as possible of the unknown quantities in terms of one of the unknowns. For instance, if one term B is twice as large as another term A, it is represented as $2A$.

Step 3.
If all of the unknown quantities cannot be expressed in terms of one of the unknowns (as suggested above) then as many equations must be written as there are different letters used to represent the unknowns. Each of these equations must represent a separate, independent relationship. After the necessary number of equations has been obtained they may be solved, simultaneously, and the values of the several unknowns determined.

SOLUTION OF EQUATIONS

After writing an equation, or having been given an equation, the next step is to find the values of the unknowns. This procedure is referred to as solving the equation, finding the roots of the equation, or finding the values of the variables. The term roots and variables have specific meanings in some instances, but in any case their numerical values if correctly obtained are the values of the unknowns. Consequently the terms roots, variables, and unknowns are used interchangeably.

Before proceeding with the solution of any algebraic equation or equations it is important to determine if the necessary amount of information is given so that a solution is theoretically possible. If no unknowns appear in the expressions being considered, then the solution is simply a process of simplification—no unknown values are to be found. However, if the value of one unknown is to be found there must be one equation involving that particular unknown and incorporating no other. If the

number of unknowns is greater than one, it will be necessary to have as many independent equations as there are unknowns to be found These equations, equal in number to the unknowns, must be solved simultaneously—a simple operation once the solution of a single equation is understood.

It is also important to know that the number of values which can be found to satisfy a single equation of one unknown will be equal to the highest power of the unknown appearing in that equation. The *power* of the unknown is the value of the exponent affixed to that variable Exponents greater than unity are termed powers; less than unity, roots The equation $4x^2 - 9x = -2$ contains but one variable since x^2 and x are the same unknown though raised to different powers. This one equation is solvable and two answers may be expected as a result

Unfortunately, the detailed description of many operations used in solving equations makes it appear that the labor involved is considerable, when actually, the method is surprisingly brief and simple It is therefore suggested that the examples given be examined, step by step, along with the descriptive notes This will assist in correlating the associated theory with its practical application After a time, short-cuts may be discovered which will make possible more direct solutions Also, it will be possible to choose the most desirable method whenever there are several which may be employed.

Valid Operations with Equations

The rules and operations (1) to (35) inclusive have been described as applying to expressions However, since equations are statements of equality between expressions, these same rules are also used in the solution of equations In addition, certain operations are usually required in which the entire equation and not just one of its expressions are involved Since equations consist of two sides separated by the equal sign, it is possible to define these additional rules and operations as applying to *both sides* of the equation, even though one of these sides is zero This emphasis on *both sides* helps to eliminate the possibility of applying a rule or operation to only one side of an equation, and leaving the other side unaffected.

In the solution of equations, the following operations may be performed without destroying the equality of the relationship.

(36) The same or equal quantities may be added to both sides of an equation

$$\begin{aligned} 3 &= 3 \\ 2 + 3 &= 3 + 2 \\ 5 &= 5 \end{aligned}$$

(37). The same or equal quantities may be subtracted from both sides of an equation

$$\begin{aligned} 3 &= 3 \\ 3 - 2 &= 3 - 2 \\ 1 &= 1 \end{aligned}$$

(38). In an equation involving addition or subtraction, any term may be transposed from one side of the equal sign to the other provided that its plus or minus sign is reversed

$$\begin{aligned} 4 + 2 &= 6 \\ 4 &= 6 - 2 \end{aligned}$$

An analysis of Rule (38) will show that it, in reality, embraces the same principles as are set forth in Rules (36) and (37).

Transposing and changing sign is often erroneously applied to equations in which the various terms appear as products or quotients. The fallacy of such an operation is apparent if a simple problem is involved:

$$6=\frac{12}{2} \qquad\qquad (2)\ (3)=6$$

$$\frac{6}{-2}=12 \qquad\qquad 3=6-2$$

$$-3\neq 12 \qquad\qquad 3\neq -4$$

Both solutions are obviously *incorrect.* The given equations were not in addition and subtraction, but involved division and multiplication respectively, and hence the rule does not apply.

(39). Both sides of an equation may be multiplied by the same or by an equal quantity.

$$3=3$$
$$(2)\ (3)=(2)\ (3)$$
$$6=6$$
$$5+2=7$$
$$2(5+2)=(2)\ (7)$$

Expanding or performing the indicated multiplication of the above example may be accomplished by either one of two methods:

$$(2)\ (5)+(2)\ (2)=14 \qquad \text{or} \qquad 2(7)=14$$
$$10+4=14 \qquad\qquad 14=14$$
$$14=14$$

(40). Both sides of an equation may be divided by the same or by an equal quantity. (Zero can never be used as a divisor as demonstrated on Page 50.)

$$8=8$$
$$8/4=8/4$$
$$2=2$$

$$4+6=10$$

$$\frac{4+6}{2}=\frac{10}{2}$$

Performing the indicated division for examples of this type may be accomplished by several methods:

$$\frac{4}{2}+\frac{6}{2}=5 \qquad \text{or} \qquad \frac{10}{2}=5 \qquad \text{or} \qquad ½\ (4+6)=\frac{1}{2}(10)$$

$$2+3=5 \qquad\qquad 5=5 \qquad\qquad 2+3=5$$

$$5=5 \qquad\qquad\qquad\qquad 5=5$$

This example as solved by the method on the right demonstrates the fact that the quotient obtained by dividing one quantity by another is the same as the product of the reciprocal of the divisor and the quantity to be divided

(41). The reciprocal may be taken of both sides of an equation

$$4/2 = 6/3 \qquad \text{or} \qquad 2 = 2$$

$$\frac{1}{4/2} = \frac{1}{6/3} \qquad \text{or} \qquad \tfrac{1}{2} = \tfrac{1}{2}$$

The reciprocal of a fraction is merely the fraction inverted, thus if the equation is given,

$$\frac{1}{x} = \frac{5}{3}$$

Then
$$\frac{x}{1} = \frac{3}{5} \qquad \text{or} \qquad x = \frac{3}{5} \qquad \text{Rule (18)}$$

The reciprocal of both sides of an equation in which one side consists of two or more fractions is found as follows

$$\frac{1}{R} = \frac{1}{a} + \frac{1}{b}$$

$$R = \frac{1}{\dfrac{1}{a} + \dfrac{1}{b}}$$

$$R = \frac{1}{\dfrac{b+a}{ab}}$$

$$R = \frac{ab}{b+a} = \frac{ab}{a+b} \qquad \text{Rule (17) or (18)}$$

(42). Both sides of an equation may be raised by any identical power. In some cases this operation may cause additional roots to be formed which will not check when substituted into the given equation. This is explained in the paragraphs entitled Radical Equations.

$$4 = 4 \qquad\qquad x^{3/2} = 8$$

$$4^2 = 4^2 \qquad\qquad (x^{3/2})^2 = 8^2$$

$$16 = 16 \qquad\qquad (x^{3/2})(x^{3/2}) = x^{6/2} = 64$$

$$x^3 = 64$$

$$x = +4$$

(43). Any identical root may be extracted from both sides of an equation. The odd roots of any number are positive, and the even roots are positive or negative according to Rules (4) and (5).

$$25 = 25 \qquad\qquad 27 = 27$$
$$\sqrt{25} = \sqrt{25} \qquad\qquad \sqrt[3]{27} = \sqrt[3]{27}$$
$$\pm 5 = \pm 5 \qquad\qquad +3 = +3$$

(44). The logarithm may be taken of both sides of an equation.

$$x = y$$
$$\log x = \log y$$

The application of this rule will be demonstrated in Section IV—Logarithms.

In calculus, the derivative and the integral of both sides of an equation may be taken.

Practical Check on Algebraic Operations

Whenever the validity of an operation in the solution of an algebraic equation is questionable, it is a good practice to set up a simple example similar to the problem involved, but using small integers in place of more complicated terms. By using values such that the correct answer is apparent by inspection, the questionable operation performed in the simple example will set forth the procedure necessarily applied in the original more complicated equation.

The several examples shown below are but a few of the many applications of this method. In choosing integers for substitution, the use of the values 1 and 2 is to be avoided unless it is definitely known that these values will not satisfy an operation which for any other integers would be an erroneous process.

Is $\sqrt{a^2+b^2} = \sqrt{a^2} + \sqrt{b^2}$

Let $a = 4 \qquad b = 3$

$$\sqrt{4^2+3^2} = \sqrt{16+9} = \sqrt{25} = \pm 5$$

Assume $\sqrt{4^2+3^2} = \sqrt{16} + \sqrt{9} = (\pm 4) + (\pm 3) = \pm 7; \pm 1$

The questionable operation is obviously incorrect.

Is $\dfrac{m}{n} = \dfrac{x}{y}$ equivalent to $(m)\,(y) = (n)\,(x)$

$$(ny)\left(\frac{m}{n}\right) = (ny)\left(\frac{x}{y}\right)$$

$my = nx$ Multiplying by (ny) Rule (39)

Also $\dfrac{6}{3} = \dfrac{4}{2}$

$$(6)\,(2) = (3)\,(4)$$
$$12 = 12$$

The questionable operation is correct. This operation is known by some as: In a proportion the product of the mean quantities is equal to the product of the extremes. This relationship may be referred to as the diagonal rule, or simply that the cross-products of a proportion are equal. (See page 35.)

Is $$\frac{\sqrt{2}}{2} = \frac{(\sqrt{2})^2}{2^2}$$

$$\frac{\sqrt{2}}{2} = \frac{1\,414}{2} = .707$$

$$\frac{(\sqrt{2})^2}{2^2} = \frac{2}{4} = .5$$

The questionable operation is obviously incorrect There is *no* rule which states that both numerator and denominator of a fraction may be squared (See Rule (42) for comparable operation involving equations).

Is $$\frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{2}\sqrt{2}}$$

$$\frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{(2)\,(1\,414)}{2} = 1\,414 \qquad \text{Rule (9)}$$

The operation as shown above is correct, being an application of Rule (9). This simplification is frequently used to change the denominator to an integer and thus a more convenient divisor

Is $$-\frac{6x-4}{2} = -3x - 2 \text{ or } -3x + 2$$

$$-\frac{6x-4}{2} = -3x + 2 \qquad \text{Rule (26)}$$

Because, Let $x = 2$

$$-\frac{6x-4}{2} = -\frac{12-4}{2} = -\frac{8}{2} = -4$$

$$-3x - 2 = -6 - 2 = -8 \qquad \text{(incorrect)}$$
$$-3x + 2 = -6 + 2 = -4 \qquad \text{(correct)}$$

Is $$(b^2)\,(b^3) = b^{(2)(3)} \text{ or } b^{2+3}$$
$$(b^2)\,(b^3) = (b^5)$$

Because, $$\text{Let } b = 2$$
$$(2^2)\,(2^3) = (4)\,(8) = 32$$
$$(2^2) = 2^{(2)(3)} = 2^6 = 64 \qquad \text{(incorrect)}$$
$$(2^2)\,(2^3) = 2^{2+3} = 2^5 = 32 \qquad \text{(correct)}$$

METHODS OF SOLUTION—SINGLE EQUATION

The solution of any algebraic equation is accomplished by an application of one or more fundamental rules which govern the operations involved. These rules may be employed according to a definite plan, in which case the procedure is called a method of solution.

In explaining the various methods of solution the term *unknown* and the expression *degree of an equation* are frequently used. These have been previously defined.

The word *coefficient,* not previously employed by name, will be used in the following pages. This term refers to the prefix of the unknown term written to indicate how many times the unknown is to be taken as a factor; thus $5x$ means 5 times (x) or the quantity $5x$. As in this example, the coefficient is usually a numerical value although not necessarily an integer. It is sometimes represented by one of the first letters of the alphabet as a, b, c.

Other numerical values in an equation unattached to unknowns are called *constant terms.* For example, $5x^2 + 10x + 15 = 20$ contains two constant terms, 15 and 20, which may be combined into a single value of ± 5 depending on which side of the equation the terms are collected according to Rule (38). The name constant is appropriate to such terms since numerical values are of constant or unvarying magnitude. In this same equation the coefficients 5 and 10 are also constants from this viewpoint, but should be considered as coefficients inasmuch as they are attached to unknowns.

Equations varying from the simple to the complex are sometimes referred to as polynomials, meaning many numbers. However, this is not the principal use of this term, since a polynomial is any algebraic expression of two or more terms, as $(x + 2)$, $(a + b + c)$. Another word, *function,* is also used to mean an equation. This latter term will be used in preference to the word polynomial when a substitute term for equation is desired. This will allow the meaning of polynomial to be restricted to the use of expressions only.

The following methods of solution are applications of the rules and operations described in the preceding pages. In many cases the answer or answers are obtained by the performance of a single operation, but to the other extreme, a great deal of computation may be required. Furthermore, for certain types of problems, the results may be obtained by following two or more distinct procedures. The applications and relative merits of each of these solutions are not all at first evident. The *best* method of solution will also be a matter of the user's personal likes and dislikes. Where a choice in the method of solution exists it is only by experience that the most convenient and practical form or forms will become apparent.

Simplification

Any equation of the first degree involving only *one* unknown appearing any number of times can be solved by means of simplification:

$$\begin{aligned} 2x - 2 &= 4x - 4 \\ +4 - 2 &= 4x - 2x \\ 2 &= 2x \qquad \text{Rule (38)} \\ x &= 1 \end{aligned}$$

For no apparent reason it is customary to transpose and collect all terms of the variables on the left side of the equal sign and to do likewise with the constant terms on the right side. There is no fallacy in this as it is possible to use either side for these terms. However, it seems logical to transpose all of the variable terms to that side where, when collected, their sum will be a positive $(+)$ quantity. In the example above this happens to be the right-hand side, and by so doing one step in the solution may be omitted. In the last step, it makes no difference if the equality is written:

$$1 = x$$

$$\text{or} \quad x = 1$$

In many cases a group of terms are included within parentheses to show that the group is to be treated as a single quantity To solve such an equation it is necessary to first remove such parentheses and consolidate the terms as much as possible.

$$x + 3(2 - a) = 2a$$
$$x + 6 - 3a = 2a$$
$$x = 2a + 3a - 6 = 5a - 6$$

Although an extremely simple example, the above equation demonstrates the fact that a parenthesis sign preceded by a positive factor has no effect upon the sign ($\pm$) of the enclosed terms when the parentheses are removed The same holds true when no term precedes the parenthesis other than the $+$ sign In this case the quantity is to be taken $+1$ times, although the number 1 is not shown, merely inferred.

The presence of either a ($-$) sign or a negative factor preceding an expression enclosed within parentheses serves to reverse the sign of each term of the expression when the parentheses are removed The procedure involved in the removal of parentheses is based directly on the rules governing the products of positive and negative numbers.

The accompanying example shows the application of these rules and also the occurrence of a parentheses within a parentheses In such cases it is recommended procedure to commence with the innermost expression first and remove one pair of parentheses at a time

$$2x - (-2a - (a - b) + b) + 4a = 7a$$
$$2x - (-2a - a + b + b) + 4a = 7a$$
$$2x + 2a + a - 2b + 4a = 7a$$
$$2x + 7a - 2b = 7a$$
$$2x = 7a - 7a + 2b = 2b$$
$$x = b$$

The solution of an algebraic equation involving fractions is generally simplified if the equation can be rearranged so that all fractions are removed and all numerical values appearing in the resulting equation are integers One method by which this can be accomplished is to first change all terms of the given equation to a common denominator, after which this denominator is discarded according to Rule (39): The value of an equation is not changed by multiplying both sides by the same or an equal quantity.

The operation as described above is in effect the same as multiplying the original equation by a quantity known as the least common multiple, abbreviated, L C. M.

(45). The value of the L C. M is the same as that of the L. C D, and is found by an identical method.

Step 1.
Find the prime factors of each of the denominators when each fraction is represented in its *simplest form* (See second example).

Step 2.
Find the product of all the different prime factors, using each factor the greatest number of times it occurs in any one denominator. This is the L. C. M. of the given equation.

$$x/2 + x/6 = 2 \quad \text{or,} \quad x/2 + x/2.3 = 2/1$$
$$\text{L. C. M.} = (2)\ (3) = 6$$
$$6x/2 + 6x/6 = 12/1$$
$$3x + x = 12$$
$$4x = 12$$
$$x = 3$$

The elimination of fractions from an equation by multiplying each term of both sides by the L. C. M. is more direct than the previously described method, consequently it should be employed in all such cases.

In determining the L. C. M. the failure to represent each fraction in its simplest form may lead to an erroneous answer as demonstrated in the following example:

$$\frac{4x - 12}{x - 3} + \frac{6}{x - 2} = 5$$

Assuming $(x - 3)\ (x - 2)$ to be the L.C.M.

$$(x - 2)\ (4x - 12) + 6(x - 3) = 5(x - 3)\ (x - 2)$$
$$4x^2 - 20x + 24 + 6x - 18 = 5x^2 - 25x + 30$$
$$x^2 - 11x + 24 = 0$$

The solution of this equation is accomplished by methods described in the solution of Quadratic Equations, a subject reserved for later study. However, according to elementary theory, any equation such as $x^2 - 11x + 24 = 0$ should have two answers for the value of x. For any given equation these values are usually different in absolute value and often times in algebraic sign as well. But, in many cases they may have identical values in every respect. Regardless of the nature or number of the answer obtained, the values resulting from the solution of the equation should satisfy the given equation when each of such numbers is substituted into the equation in place of the unknown quantity. If the answer obtained fulfills this requirement, it can be called a *root* of the equation. The distinction between a root and an answer is that a root is an answer known to be correct for the given equation. Many answers, even though determined by correct algebraic methods, will not be roots of the given equation.

The roots of the equation $x^2 - 11x + 24 = 0$ are found to be 3 and 8. The solution of this equation is accomplished by methods considered later, but the validity of the roots can be easily ascertained. However, 3 cannot be a root of the given equation,

$$\frac{4x - 12}{x - 3} - \frac{6}{x - 2} = 5$$

as it makes the first fraction zero and the value of the second fraction -6 which destroys the equality of the relationship. The equation $x^2 - 11x + 24 = 0$ is, therefore, not equivalent to the given equation. An inspection of the first fraction of the given equation reveals that it is not in its simplest form since

$$\frac{4x - 12}{x - 3} \text{ is } \frac{4\ (x - 3)}{x - 3} \text{ or } 4.$$

Therefore, the L.C.M. is $(x - 2)$, and the simplified equation is:

$$4(x-2)+6=5(x-2)$$
$$4x-8+6=5x-10$$
$$10-8+6=5x-4x$$
$$8=x$$

By substitution 8 is found to be a root of the given equation. Note that the unknown quantity in this case is in the denominator, and that this is the only place where it occurs

The importance of using each fraction in its simplest form when determining the L.C.M for the given equation is apparent from the example above From this it might be implied that the use of the L.C.M is essential to the solution of equations involving fractions This is not true, however, because in most cases the roots of the equation resulting from clearing the original equation of fractions are the same as the roots of the original equation, whether the L.C.M. is used or not This will always be true when no denominator of the original equation contains the unknown. However, the elimination of fractions from an equation by multiplying each term of both sides by the L.C.M is the most direct solution and should be employed in all cases.

When the variable appears in the denominator of one or more of the terms of the original equation, it is quite possible that the equations formed by clearing the original equations of fractions by multiplying each term by the L.C.M., will not be equivalent to the given equation, and the resulting equation may have roots that are not roots of the given equation Such roots which are found by correct algebraic means, yet will not check in the original equation are called *extraneous roots* It is, therefore, necessary to determine by substitution whether any roots of this simplified equation are not roots of the original equation If they are not, this fact is at once apparent, for if such a root is substituted in the original equation it will make some denominator of the original equation equal to zero A root of this kind cannot be a root of the original equation

The simplification of some equations involving fractions may lead to an equation of the form

$$\frac{2}{3}x=6$$

Assuming that it is the value of (x) that is desired, the above equation can immediately be solved by multiplying both sides of the equation by 3/2 or the reciprocal of the coefficient of the unknown term

$$\left(\frac{3}{2}\right)\left(\frac{2}{3}x\right)=\left(\frac{3}{2}\right)(6) \qquad \text{Rule (39)}$$
$$x=9$$

> To find the reciprocal of any number simply divide one by that number The reciprocal of a fraction is merely the fraction inverted Thus the product of any number and its reciprocal is always $+1$

The use of the above method in preference to the use of the L.C.M. is left to the user's discretion. Both methods are correct

Equations taking the form of proportions (described in the paragraphs titled Ratio, Proportion and Variation) are so frequently encountered that it is advisable to show

by an example the most practical solutions. The position of the unknown, x, may be in any one of the four possible positions.

$$\frac{a}{b}=\frac{x}{c}$$

$$\frac{ac}{b}=x \text{ Multiplying by } (c) \qquad \text{Rule (39)}$$

Another solution involving one additional step yet more commonly used than the above makes use of the fact that the cross-products of a proportion are equal. This relationship may be referred to as the diagonal rule. This operarion in Geometry is stated: In a proportion the product of the mean quantities is equal to the product of the extremes.

$$\frac{a}{b}=\frac{x}{c}$$

$$bx=ac$$

$$x=\frac{ac}{b}$$

Multiplying by (c) and (b)
at the same time. Rule (39)

An equation containing only one unknown with that unknown occurring only once and with an exponent other than unity (either a root or a power) can be solved by changing the entire equation to the desired power of the unknown. Assuming that it is the value of x^1 that is desired:

$$x^2=16$$
$$x=\sqrt{16}=\pm 4 \text{ Rule (37)}$$

$$x^{3/2}=8$$
$$(x^{3/2})^{2/3}=8^{2/3}=(8^2)^{1/3}$$
$$x^{6/6}=(64)^{1/3}$$
$$x=\sqrt[3]{64}=+4$$

$$3x^3=64+2x^3$$
$$x^3=64$$
$$x=+4$$
$$\text{or } x^{3/2}=8$$
$$(x^{3/2})^{2/3}=8^{2/3}=(8^{1/3})^2$$
$$x^{6/6}=(2)^2$$
$$x=4$$

The use of the fact that the product of a fraction and its reciprocal is always $+1$ is of distinct advantage in the solution of this problem. That this is true is apparent if the solution of the same problem by another method is examined.

Many equations include a number of terms arranged to indicate that more than one operation is to be performed. An unlimited number of such examples can be shown, but only two arbitrarily chosen types will demonstrate the fact that only basic principles included in Rules (1) to (45) inclusive are involved.

The equation ($L=C\frac{r}{2}SV^2$), solved for V, presents a type of solution frequently required in practical work.

$$2L = CrSV^2$$

$$\frac{2L}{CrS} = \frac{CrSV^2}{CrS} \qquad \text{Rule (40)}$$

$$\frac{2L}{CrS} = V^2 \qquad \text{Rule (10)}$$

The three factors C, r and S by which both sides of the equation are being divided are not ordinarily written down as a denominator on the side of the equation from which these terms are to be eliminated. This is obviously a waste of effort as the quotient in such cases will always be 1. However, in the example above, this step has been performed to emphasize that to isolate the unknown term from any factors which may accompany it, simply divide both sides of the equation by these factors.

$$\frac{2L}{CrS} = V^2 \qquad \text{Rule (10)}$$

$$V = \sqrt{\frac{2L}{CrS}} \qquad \text{Rule (43)}$$

The second example is of a more complex nature. Fortunately such types are infrequently encountered.

$$a = \frac{b}{1 + \left(\frac{18}{R}\right) b}$$

For purposes of explanation, assume that R is the unknown quantity, and that (a) and (b) are arbitrary constants to which any one of an unlimited set of numerical values may be assigned.

The value of R is then to be determined in terms of (a) and (b). The solution as given is not necessarily written in its briefest form, as it is intended to clearly show each operation involved.

$$a = \frac{b}{1 + \frac{18b}{R}} \qquad \text{Rule (16a)}$$

$$a = \frac{b}{\frac{R + 18b}{R}} \qquad \text{Rule (9), (11)}$$

$$\frac{a(R + 18b)}{R} = b \qquad \text{Rule (39)}$$

$$\frac{R + 18b}{R} = \frac{b}{a} \qquad \text{Rule (40)}$$

$$1 + \frac{18b}{R} = \frac{b}{a}$$

$$\frac{18b}{R} = \frac{b}{a} - 1 = \frac{b - a}{a} \qquad \text{Rule (11), (38)}$$

This solution may be completed by alternate methods.

$$R\,(b-a)=18ab$$

$$R=\frac{18ab}{b-a} \qquad \text{Rule (40)}$$

$$\frac{R}{18b}=\frac{a}{b-a} \qquad \text{Rule (41)}$$

$$R=\frac{a\,(18b)}{b-a} \qquad \text{Rule (39)}$$

$$R=\frac{18ab}{b-a}$$

Trial and Error

This method, which in a few cases is the most rapid solution, is limited in its application to simple equations, or to more involved equations for which only an approximate answer is required. Trial and error methods are also used in establishing the factors of an equation as explained in the paragraphs entitled Factoring.

Trial and error consists of assigning an arbitrary number to the unknown and determining, by substitution, if the assumed number is a root of the equation. A root of an equation is a number which when substituted for the unknown quantity will make the two sides of the equation equal, that is, it satisfies the equation.

The work involved in the solution of an equation by this method is reduced if the given equation is first simplified. Furthermore, it is advisable to transpose all terms of the equation to one side of the equal sign with the other side of the equation becoming zero.

$$5x^2+10x+15=30$$

$$x^2+2x+3=6 \qquad \text{Rule (40)}$$

$$x^2+2x-3=0 \qquad \text{Rule (38)}$$

The given equation is now expressed in its simplest form and with all terms transposed to one side of the equation. For the purpose of explanation, and not necessarily the most logical procedure, the value of the unknown is first assumed to be 2.

$$\begin{aligned}(2)^2+2(2)-3&=0\\ 4+4-3&=0\\ 5&=0\\ x&\neq 2\end{aligned}$$

Assume $x=3$:

$$\begin{aligned}(3)^2+2(3)-3&=0\\ 9+6-3&=0\\ 12&=0\\ x&\neq 3\end{aligned}$$

As the error is increasing with increasing values assumed for (x), it is reasonable to assume that the value of the unknown is smaller and not greater than the first assumed value for the unknown which was 2. This reasoning is not always valid as may be seen by an examination of the curve of the second example in the paragraphs entitled Graphical Solution, page 42.

Assume $x = 1$.

$$\begin{aligned} (1)^2 + 2(1) - 3 &= 0 \\ 1 + 2 - 3 &= 0 \\ 0 &= 0 \\ x &= 1 \end{aligned}$$

The given equation is of the second degree and consequently has two values to be determined It is logical to assume that this second value is a negative number since it was shown above that the larger the value assumed for (x), the greater the magnitude of error Therefore, the next value arbitrarily assumed for the unknown is -2.

$$\begin{aligned} (-2)^2 + 2(-2) - 3 &= 0 \\ 4 - 4 - 3 &= 0 \\ -3 &= 0 \\ x &\neq -2 \end{aligned}$$

It is apparent that $x = -2$ is not a root of the equation An inspection of the equation shows that any negative value substituted for the unknown becomes positive for the x^2 term and remains negative in the x term The constant -3 is unaffected by values assumed for the unknown Since the value of the equation is negative for a value of $x = -2$, a larger negative value or $x = -3$ will be assumed next because such a number will make the equation as a whole increase in positive value.

$$\begin{aligned} (-3)^2 + 2(-3) - 3 &= 0 \\ 9 - 6 - 3 &= 0 \\ 0 &= 0 \\ x &= -3 \end{aligned}$$

The two roots of the given equation,

$$5x^2 + 10x + 15 = 30$$

or its simplified equivalent,

$$x^2 + 2x - 3 = 0$$

are, therefore,

$$x = 1 \text{ and } x = -3$$

Additional examples involving the use of the trial and error method might be given, but they would only show the tediousness of this procedure because the variation of the value of the equation with the variation in the value assumed for the unknown is difficult to predict in many cases The graphical solution, to be described in the following paragraphs, shows the variation of the value of the equation as the value of the unknown varies, and in this respect alone justifies its preference to the trial and error method. The graphical solution has an additional benefit in that it indicates approximately the value of any irrational roots which may exist, a feature which becomes exceedingly laborious using trial and error.

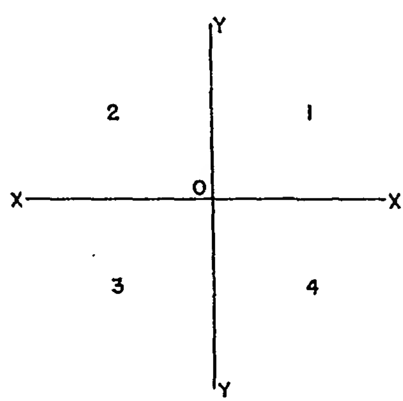
Graphical Solution

There is no elementary algebraic method of solving equations of the third or higher degree unless some of the roots may be discovered by trial and error or by factoring, as explained in the paragraphs bearing these titles The application of either of these methods from a practical viewpoint is limited, and altogether impossible, in the case of factoring, if irrational roots are involved However, any equation in any degree, as

long as only one unknown is involved, may have its real roots approximated by a simple although somewhat tedious graphical method. The term real root as introduced here is the same as the term used before simply as root. The term *real* differentiates such roots from *imaginary* roots. By an imaginary root or an imaginary number is meant a number involving the square root, or any even root, of a negative number. For ordinary mathematical purposes it is assumed that the square root of a negative number does not exist. (For actual facts, investigate complex numbers in advanced algebra.) If any of the roots of an equation are imaginary, they will not appear on a graph plotted of that equation.

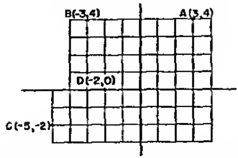
All algebraic equations can be represented by a line or a curve plotted on a plane. Such a line or curve is established by first finding a number of points, each of which is known to be one point through which the curve of the equation must pass. A number of such points are plotted and through these the curve or line may be drawn which is called the curve or the graph of the equation.

In graphical representation, the plane upon which a point may be located is divided into four zones, or quadrants, which are numbered in a counter-clockwise direction with the upper right-hand quadrant as number one.



The lines separating the quadrants are termed *axes*; the vertical line is designated as the *Y-Y* axis, and the horizontal line as the *X-X* axis. The point of intersection of these two axes is termed the *origin*, or zero point, and from this point all measurements of distance are made. Distance along, or parallel to, the *X-X* axis is termed the *abscissa*, and is considered positive if measured to the right of the *Y-Y* axis and negative if measured to the left. Distance along, or parallel to, the *Y-Y* axis is termed the *ordinate*, and is positive if measured above the *X-X* axis, and negative if measured below.

If a point lies in a plane, its location may be determined by two measurements of distance, or coordinates, one showing its distance from the *Y-Y* axis, (abscissa), and the other its distance from the *X-X* axis, (ordinate). It is customary to write the coordinates of a point in parenthesis with the abscissa first. Thus the plotting of $A\ (3,4)$, $B\ (-3,4)$, $C\ (-5,-2)$, and $D\ (-2,0)$ places the points in the positions and quadrants as shown below.



The above system of locating a point on a plane is known as Cartesian coordinates. A second method of locating a point in a plane is described in Section III—Trigonometry. The graphical solution of an equation of the type named above consists of arbitrarily assigning numerical values to the unknown and computing the corresponding values of the equation The values assumed for the unknown are plotted as abscissas and the corresponding values of the equation as ordinates

Each pair of values so determined establishes a point and through a number of such points a smooth curve can be drawn. The intersection of this curve with the X-X axis establishes a value of the abscissa, which is a root of the given equation

In solving equations by the graphical method, the term *function* is used to replace the word equation, and the term *variable* is used in place of unknown. This terminology is appropriate inasmuch as the equation is a function of the unknown since it depends on this latter quantity for its value To find the value of an equation it is necessary to transpose all of its terms to one side of the equal sign Consequently, the term function implies an equation in which all terms are arranged in this manner. In the study of functions the unknown is often called the variable, since from this point of view the problem is not so much the finding of a value for the unknown as it is the study of the changes of a variable quantity Thus the terms function and variable more clearly express the relationship of the unknown to the entire equation than the words formerly employed

The foregoing explanation may be condensed into several distinct operations and thus clarify the actual procedure to be followed.

Step 1.
Arrange all terms of the given equation on one side of the equal sign and simplify the resulting expression, now termed the function.

Step 2
Arbitrarily assign different values for the variable and determine the corresponding values of the function Plot a number of points corresponding to the number of pairs of values determined in this manner, using the value of the variable as abscissa and the corresponding value of the function as ordinate Obviously, if the value of the function becomes zero with some assumed value for the variable, then that value of the variable is a root of the equation

Step 3.
Through the number of points thus established draw a continuous smooth curve. The value of the abscissa of any point at which this curve

either crosses or is tangent to the X-X axis is the value of a root of the given equation. This fact may be verified by substituting the value found into the given equation.

The graphical method of solving any equation in any degree as long as only one unknown is involved, is nothing more than a pictorial representation of the trial and error solution in which the value of the equation, with all terms on one side of the equal sign, becomes zero upon assuming a value of the unknown corresponding to a root.

The examples on page 42 illustrate the method of graphical solution. The equation used for the first example is of the second degree and has roots which are small integers. Its roots have been already found by trial and error. In the second example the usefulness of the graphical solution is more apparent. In both examples the total number of possible roots have been found because in each case the curve crosses the X-X axis the number of times which corresponds to the degree of the equation. Consequently no imaginary roots exist for these selected equations.

It might be suggested at this point that a combination of the trial and error method together with the graphical method is a convenient means of finding either fractional or irrational roots. First use the trial and error method to find the approximate or boundary roots between which the precise root is known to exist, and then, by the graphical method, plot only this portion of the curve using several points adjacent to the axis. The greater the number of points employed, the more exact the curve is represented, and hence the more accurate the approximation will be.

In any graphical representation or solution, the scale of the drawing will be the controlling factor, not considering errors in solution, in the accuracy of the result. Obviously the larger the scale of the drawing, the more accurate can be the solution. For this reason two alternatives are employed to effect more accurate results.

One of these alternatives was applied in the solution of example 2, and consists of using different scales for ordinate and abscissa in order that it may be easier to accurately estimate the point at which the curve crosses the X-X axis.

The second alternative is to plot a second curve, or set of curves, involving only those portions of the original curve which cross the X-X axis. These portion-curves should be drawn through several points located adjacent to the X-X axis, and to a scale several times as large as that of the original continuous curve from which the approximate points of intersection were discovered.

The application of these two expedients introduces no operation other than those already explained.

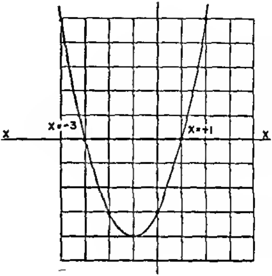
There are several characteristics of the curves of equations which should be kept in mind when solving such equations by the method of plotting successive points. These are not peculiarities of all equations, but must be recognized when encountered.

The principle involved in graphically solving algebraic equations is that if the curve of the equation lies above the X-X axis for one value of the unknown and below for another value, there must be a root of the equation somewhere in between. This method is very effective when used in conjunction with certain methods of differential calculus. However, when the only means of obtaining the curve is to plot successive points, the procedure is sometimes very misleading. This is true because it is not always possible to determine the exact shape of the curve near the critical values.

Not demonstrated in the previous examples is the occurrence of *multiple roots* de-

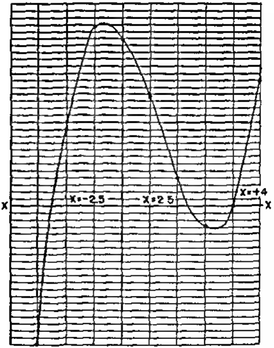
Example 1: $5x^2 + 10x - 15 = 0$
or $x^2 + 2x - 3 = 0$ (simplified)

LET X =	VALUE OF FUNCTION
−5	+12
−4	+ 5
−3	0
−2	−3
−1	−4
0	−3
1	0
2	+5
3	+12
4	+21



Example 2 $x^3 - 4x^2 - 6.25x + 25 = 0$

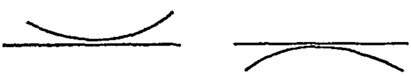
LET X =	VALUE OF FUNCTION
−3	−19.75
−2	+11.00
−1	+26.75
0	+25.00
1	+15.75
2	+4.50
3	−2.75
4	0.00
5	+16.75



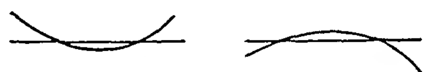
fined as two or more identical roots appearing in the same equation. If any equation containing multiple roots is plotted, the curve will be tangent (touch but not cross) to the X-X axis at a point whose abscissa is the value of the multiple roots. For every such point of tangency it should be considered that two roots of the equation have been determined.

Another factor of importance concerns the plotting of functions in which one or more fractions exist with the variable included in the denominators of one or more of such terms. In such cases the curve will not be continuous, but will break if such values of the variable are assumed which will make the denominator become zero with the numerator some other value.

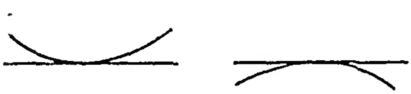
The following sketches illusttate examples where the graphical method of solution either breaks down completely or must be used with great caution.



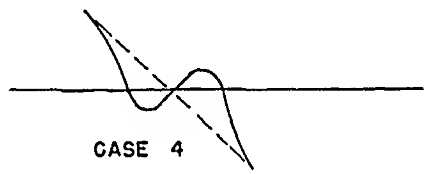
CASE 1



CASE 3



CASE 2

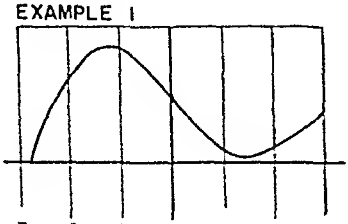


CASE 4

In Case 1 the curve approaches the axis closely, but does not quite touch it. In Case 2 it is tangent to the axis, and in Case 3 it crosses the axis at two near-by points.

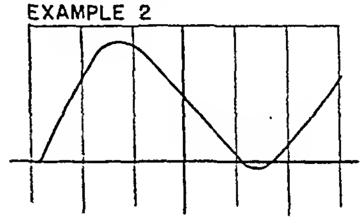
Case 1 can be detected only by advanced methods such as are used in solving the equation of Example 1 shown below. Similarly, Case 2 can be detected only by advanced methods, unless the point of tangency is located by guessing the exact value for the variable. This is unlikely unless some of the roots are small integers. The condition represented by Case 3 can be detected by plotting points sufficiently close together, but is easily overlooked in practice as can be seen in Example 2 below, that is, it may be mistaken for Case 1, and two roots thereby lost. In Case 4 the dotted curve is easily mistaken for the true one so that only one root is believed to exist when actually there are three. This can also be avoided by plotting points sufficiently close together.

EXAMPLE 1



$3X^3 - 18X + 17 = 0$

EXAMPLE 2



$3X^3 - 18X + 16 = 0$

The bend points of both graphs can be shown to be at $x = -\sqrt{2}$ and $x = +\sqrt{2}$. The curve in Example 1 misses the axis by approximately .03. The curve in Example 2 crosses the axis between 1.1 and 1.2 and again between 1.6 and 1.7. Since the function

of $3x^3 - 18x + 16$ is positive for both $x = 1$ and $x = 2$, the roots between 1 and 2 might both be lost

Factoring

Factors have been previously defined as numbers that are multiplied together, and also as quantities which can be exactly divided into the given expression The process of factoring is a reverse process to that of multiplication, for in factoring one starts with a product and endeavors to find the factors which have been multiplied together to produce the given quantity When the given quantity has been completely factored, the results obtained are called *prime factors* Such factors are easily recognized since they are exactly divisible only by themselves or by one Not all quantities are factorable as they may already be in the form of a prime factor, even though the given quantity may consist of several terms and consequently appear to be a product of two or more simpler terms.

The first step, and one of several operations involved in the method of solving an equation by factoring, consists of transposing all terms to one side of the equal sign. The terms are at the same time arranged so that they are in an order in which the factors may be discovered by inspection in some simple cases, or by some method of computation for more complex problems as explained later. Each factor thus found is further factored, if possible, so that the prime factors are obtained The prime factors are then written in place of the given equation producing an *equivalent* though different appearing relationship. For example in the equation $x^2 + 2x - 3 = 0$, the factors are $(x + 3)$ and $(x - 1)$ as shown on page 42 The partial solution of the equation is.

$$x^2 + 2x - 3 = 0$$
$$(x + 3)\,(x - 1) = 0$$

An analysis of the equation as represented by the product of its prime factors shows that one factor multiplied by another factor can produce zero as a product only (*a*) when one of the factors itself is zero, or (*b*) when both of the factors are zero. This fact provides a means of solving the equation, because if·

$$x + 3 = 0, \text{ then } x = -3, \text{ and if}$$
$$x - 1 = 0, \text{ then } x = +1.$$

Thus the roots of the equation are $x = -3$ and $x = +1$. The validity of these roots is established by substituting them into the given equation.

$$(-3)^2 + 2(-3) - 3 = 0$$
$$9 \quad - 6 \;- 3 = 0$$
$$0 = 0 \qquad \text{Check}$$
$$1 + 2 - 3 = 0$$
$$0 = 0 \qquad \text{Check}$$

It is advisable at this time for the user of this method of solution to check himself in regard to the definitions of the terms factors and roots, as these two words are often confused. In the example above the factors were stated to be $x + 3$ and $x - 1$. The roots were found to be $x = -3$ and $x = +1$ respectively. From these relationships an important rule can be formulated which, for convenience, is stated in the reverse order to that just given

(46). For any equation, of any degree, if n is found to be a root of that equation, then $(x - n)$ is a factor.

The prime factors of many algebraic equations of the second degree are found by the trial and error method simply by finding two factors which when multiplied together will produce the given equation as a product. Since the factors of such equations are usually polynomials it is essential to understand a method of multiplying polynomials together with a minimum of effort. The rule is as follows:

(47). The product of two polynomials is equal to the algebraic sum of the products obtained when all terms of one polynomial are multiplied by each term of the other polynomial.

The application of this rule is demonstrated by two equivalent methods with the recommendation that the first form be employed until experience justifies the use of the second method.

$$(3-2)\,(4+5) =$$

$$\begin{array}{lllllll}
 & 3-2 & & & & & \\
 & \underline{4+5} & & & & & \\
(3)\,(4) & - & (2)\,(4) & & & & \\
 & & & + & (3)\,(5) & - & (2)\,(5) \\
\hline
12 & - & 8 & + & 15 & - & 10 \quad = 9
\end{array}$$

$$(3-2)\,(4+5) = 9$$

$$\begin{array}{l}
(x+3)\,(x-1) = \\
\quad x+3 \\
\quad \underline{x-1} \\
x^2+3x \\
\quad \underline{\quad -x-3} \\
x^2+2x-3
\end{array}$$

$$(x+3)\,(x-1) = x^2+2x-3$$

In computing the product of two or more polynomials, the products of terms involving the same variables with the same exponents are listed in columns as shown. This simplifies the operation of collecting terms.

$$\left(\frac{b^2}{3} - \frac{b}{2} - \frac{1}{2}\right)\left(\frac{b^2}{2} + \frac{b}{5} + \frac{1}{2}\right) =$$

To facilitate solution only, and not a necessary operation, *each* polynomial is reduced to *its* least common denominator. Since the denominator of all products will then be 60, they are not written down in each instance. However, this denominator must be included in the answer.

$$
\begin{array}{r}
\frac{2b^2}{6} - \frac{3b}{6} - \frac{3}{6} \qquad\qquad\quad \\
\frac{5b^2}{10} + \frac{2b}{10} + \frac{5}{10} \qquad\qquad\quad \\
\hline
10b^4 - 15b^3 - 15b^2 \qquad\qquad\quad \\
4b^3 - \ 6b^2 - \ 6b \qquad\quad \\
10b^2 - 15b - 15 \\
\hline
10b^4 - 11b^3 - 11b^2 - 21b - 15
\end{array}
$$

$$\left(\frac{b^2}{3} - \frac{b}{2} - \frac{1}{2}\right)\left(\frac{b^2}{2} + \frac{b}{5} + \frac{1}{2}\right) = \frac{10b^4 - 11b^3 - 11b^2 - 21b - 15}{60}$$

The second method of multiplying factors together is the same as that already shown with the elimination of the tabulation of products.

$$(x+3)(x-1) = x^2 - x + 3x - 3 = x^2 + 2x - 3$$

This procedure can be used in any given problem but becomes of less practical value in finding the products of polynomials if each polynomial consists of more than a few terms

Polynomials involving radical signs are solved as follows:

Example 1.

$$
\begin{aligned}
(\sqrt{x-1})\,(\sqrt{x-1}) &= (\sqrt{x-1})^2 = (x-1)^{1/2}\,(x-1)^{1/2} \\
&= (x-1)^{2/2} = x-1
\end{aligned}
$$

Example 2. $\qquad (1+\sqrt{x+3})^2 \quad =$

$$
\begin{array}{l}
1 + \sqrt{x+3} \\
\underline{1 + \sqrt{x+3}} \\
1 + 1\sqrt{x+3} \\
\qquad \underline{1\sqrt{x+3} + (x+3)} \\
1 + 2\sqrt{x+3} + \ x+3
\end{array}
$$

or, $\quad x + 2\sqrt{x+3} + 4$

Factors of some equations can be found by comparing the given equation with several type forms for which the factors are known In these equations the letters of the alphabet, (a), (b), and (c), are arbitrary constants corresponding to the numerical coefficients of the actual equation.

1. $\pm ax \pm ay \pm az = a(\pm x \pm y \pm z)$
2. $x^2 - b^2 = (x-b)\,(x+b)$
3. $x^2 + 2bx + b^2 = (x+b)\,(x+b) = (x+b)^2$
4. $x^2 - 2bx + b^2 = (x-b)\,(x-b) = (x-b)^2$
5. $x^2 \pm (b+c)\,(x) \pm bc =$ (see below)
6. $x^3 + b^3 = (x+b)\,(x^2 - bx + b^2)$
7. $x^3 - b^3 = (x-b)\,(x^2 + bx + b^2)$

The forms 3 and 4 listed above are included in type 5 Equations of this type are

termed quadratic equations or equations of the second degree. The equations $x^2 = 16$ and $x^2 - 4x = 0$ are also quadratic, but are excluded from this discussion as these types are more easily solved by rules (43) and (40) respectively. Bearing in mind that the proper sign $\pm$ must be considered, the values to be assigned as the second term in each factor can be determined as follows:

> The last term, bc, of the equation is the algebraic product of the second terms of the factors, and, at the same time, the coefficient of the x term is the algebraic sum of the second terms of the factors.

Using the equation $x^2 + 2x - 3 = 0$, from the previous paragraph, and by the application of the rule as just stated, the factors, $x + 3$ and $x - 1$, can be written, since $+3$ and -1 are the only two numbers which will give -3 as a product and $+2$ as their sum.

An equation of the form $ax^2 \pm bx \pm c = 0$ can be factored by the same method as above if the equation is first divided through by the coefficient of the x^2 term, represented in the equation by (a). However, in some cases it is more convenient to factor these equations directly from the form in which they are given by the application of certain apparent relationships.

$$2x^2 + 7x + 3 = 0$$
$$(2x + 1)\ (x + 3) = 0$$

> The product of the first terms of the factors is the first term of the given equation.
>
> The sum obtained by adding the product of the two end terms of the factors to the product of the two nearest terms is the middle term of the given equation.
>
> The products of the last terms of the factors is the third term of the given equation.

$$(2x + 1)\ (x + 3) = 0$$
$$2x + 1 = 0 \qquad\qquad x + 3 = 0$$
$$2x = -1$$
$$x = -\tfrac{1}{2} \qquad\qquad x = -3$$

The solution of equations by factoring is an important method and is applicable to equations in any degree provided that such equations are factorable. In practice, however, factoring is subject to limitations because there are so many equations whose factors cannot be found conveniently, if at all. This is true where the roots of the equation are fractional or irrational. However, many equations of the third and higher degree are factorable and the explanation to follow will demonstrate the method of their solution which will include an explanation of the division of polynomials, an operation usually required in the solution of such equations.

In the previously described method for finding the factors of an equation no mention was made of finding the roots of the given equation by trial and error and from these determining what the factors would be according to Rule (46):

> In any equation, of any degree, if (n) is found to be a root of that equation, then $(x - n)$ is a factor.

This was purposely omitted in the earlier paragraphs to preclude the possibility of yielding to the temptation of working the problem in the reverse order, that is, deter-

mining the roots and then writing the corresponding factors However, for equations of higher degree such a procedure is an essentiality in order to determine as many of the roots as possible For the purpose of explanation, assume that the equation $x^4 + 4x^3 - 5x^2 + 36x - 36 = 0$ is to be factored By arbitrarily assuming several small integers as values of the variable, one of the roots of the equation is discovered, most likely that $x = -2$ Further application of the trial and error method of solution in this particular example would disclose other roots, but as a general procedure, it is advisable to simplify or factor the given equation after each root is discovered. Since $x = -2$ is known to be a root of the equation, then $(x + 2)$ is a factor, or:

$$(x + 2)\ (\text{Product of other factors}) = x^4 + 4x^3 - 5x^2 + 36x - 36$$

The value of the product of the other factors can be determined by dividing the given equation by the known factor $(x + 2)$ This is explained by steps so that the procedure as outlined may serve as an explanation for the division of any polynomial by another.

Step 1.
Arrange the terms of the expression to be divided and the divisor so that the exponents of each are in a descending order When the division is exact, that is, without remainder, the terms of both dividend and divisor can also be arranged so that the exponents of each are in an ascending order

$$x + 2\,\overline{)\,x^4 + 4x^3 - 5x^2 - 36x - 36}$$

Step 2.
Write as the first term in the quotient the result obtained when the first term of the dividend is divided by the first term of the divisor.

$$\begin{array}{rl} & x^3 \\ x + 2 & \overline{)\,x^4 + 4x^3 - 5x^2 - 36x - 36} \end{array}$$

Step 3.
Multiply the divisor by the first term in the quotient obtained in Step 2 and write the product under the dividend placing the terms containing like exponents under the corresponding terms of the dividend.

$$\begin{array}{rl} & x^3 \\ x + 2 & \overline{)\,x^4 + 4x^3 - 5x^2 - 36x - 36} \\ & \underline{\;\;x^4 + 2x^3} \end{array}$$

Step 4.
Subtract the product obtained in Step 3 from the dividend and to the remainder annex additional terms taken from the dividend so that there are in all a number of terms equal to that of the divisor.

$$\begin{array}{rl} & x^3 \\ x + 2 & \overline{)\,x^4 + 4x^3 - 5x^2 - 36x - 36} \\ & \underline{\;\;x^4 + 2x^3} \\ & \qquad\;\; 2x^3 - 5x^2 \end{array}$$

Step 5.
Write as the second term in the quotient the result obtained when the first term of the expression obtained in Step 4 is divided by the first term of the divisor.

$$
\begin{array}{r|l}
 & x^3 + 2x^2 \\ \hline
x + 2 & x^4 + 4x^3 - 5x^2 - 36x - 36 \\
 & \underline{x^4 + 2x^3} \\
 & \quad\;\; 2x^3 - 5x^2
\end{array}
$$

Step 6.
By an operation similar to Step 3, multiply the divisor by the second term in the quotient obtained in Step 5, and write the product under the dividend, placing the term containing like exponents under the corresponding terms of the dividend.

$$
\begin{array}{r|l}
 & x^3 + 2x^2 \\ \hline
x + 2 & x^4 + 4x^3 - 5x^2 - 36x - 36 \\
 & \underline{x^4 + 2x^3} \\
 & \quad\;\; 2x^3 - 5x^2 \\
 & \quad\;\; \underline{2x^3 + 4x^2}
\end{array}
$$

Step 7.
Continue as in Steps 4, 5, and 6 until all terms of the dividend have been used. In the above example this requires six more operations, and the complete solution appears as below:

$$
\begin{array}{r|l}
 & x^3 + 2x^2 - 9x - 18 \\ \hline
x + 2 & x^4 + 4x^3 - 5x^2 - 36x - 36 \\
 & \underline{x^4 + 2x^3} \\
 & \quad\;\; 2x^3 - 5x^2 \\
 & \quad\;\; \underline{2x^3 + 4x^2} \\
 & \qquad\quad - 9x^2 - 36x \\
 & \qquad\quad \underline{- 9x^2 - 18x} \\
 & \qquad\qquad\quad - 18x - 36 \\
 & \qquad\qquad\quad \underline{- 18x - 36}
\end{array}
$$

Therefore: $(x + 2)\ (x^3 + 2x^2 - 9x - 18) = 0$

A comparison of the quotient $x^3 + 2x^2 - 9x - 18$ with the type forms of cubic equations which are factorable by inspection indicates that an expression of this form cannot be factored by this method. Any further factoring must be accomplished by other means. The trial and error method may be employed to find a second root of the equation by finding some value of the variable which will satisfy either the original equation or the expression remaining to be factored. The same numerical value will satisfy both of these. Consequently, the factor having the fewer number of terms should be investigated.

An investigation of the expression $x^3 + 2x^2 - 9x - 18$ shows that $x = 3$ is a root and $(x - 3)$ is a factor of this expression and, therefore, of the original equation as

well. Consequently, a choice exists between dividing the original equation by the product of $(x+2)$ and $(x-3)$ or to divide the expression $x^3+2x^2-9x-18$ by $(x-3)$. The latter procedure is adopted and is accomplished by the method previously described for the division of one polynomial by another. The result obtained from this division is x^2+5+6

$$\text{Therefore,}\quad (x+2)(x-3)(x^2+5x+6)=0$$

The expression x^2+5x+6 is a quadratic of the type $x^2+bx+c=0$ and for the values $b=5$ and $c=6$, the prime factors $x+2$ and $x+3$ are obtained.

$$\text{Thus,}\quad (x+2)(x-3)(x+3)(x+2)=0$$

$$x=-2;\ +3;\ -3;\ -2$$

The presence of four integral roots as the result of the above somewhat laborious process might lead to the conclusion that such equations should be completely solved by the method of trial and error. This may be true for the example as given, but such convenient equations are not likely to occur often in any actual problem. In this instance the equation was purposely constructed so as to have integers for roots suitable for a rapid solution. Furthermore, the existence of multiple roots as in this example defies a solution by trial and error. Later examples in the paragraphs titled Equations of Higher Degree Solved as Quadratics further justifies the factoring method of solution for such equations.

In the solution of equations by the factoring method, the use of a factor having a numerical value of zero results in either an erroneous or an incomplete answer. Rule (40) states that both sides of an equation may be divided by the same or by an equal quantity provided that quantity is not zero. This qualification, that the divisor must not be zero, seems illogical when it is considered that this same quantity is so frequently used as a factor in the reverse operation of multiplication. However, it is easy to realize that any number taken zero times is still zero, or

$$(50)(0)=0$$

But if the comparable equation involving division is examined,

$$\frac{50}{0}=$$

the answer to be assigned as a quotient is highly imaginative as no number can be written as a quotient, which, when multiplied by the divisor (0) will again produce the number 50. Thus it seems best to state it as a fact that zero cannot be employed as a divisor, rather than to say that the quotient is some indefinite quantity such as infinity.

The fallacy of dividing by zero is apparent in the examples below.

Given,

$$a=b$$

$$a^2=ab$$

$$a^2-b^2=ab-b^2$$

$$(a-b)(a+b)=b(a-b)\quad \text{Dividing by } (a-b) \text{ which is } 0$$

$$a+b=b$$

But $a=b$, so,

$$2b=b$$

$$2=1$$

Given, $(2x+1)(x+3)=x^2-9$

$$(2x+1)(x+3)=(x-3)(x+3)$$

$$2x+1=x-3 \quad \text{Dividing by } (x+3)$$

$$x=-4$$

An inspection of the original equation shows that it is of the second degree and consequently has two roots.

Since only one root results from the solution as performed, it is possible that the quantity $(x+3)$ used as a divisor may be zero, and hence the operation incorrect. If the divisor $(x+3)$ is equal to zero, then (x) is equal to -3. Substituting this value into the equation proves $x=-3$, a root of the equation. Consequently the divisor $(x+3)$ was equal to zero, and the erroneous operation performed resulted in the loss of one of the roots.

It is recommended that before any equation is ever simplified by dividing each term of that equation by a common factor involving the variable, the value of that factor first be proved to be other than zero. This is easily done as described in the previous paragraph. If the value of the factor is not zero, then it may be used as a divisor according to Rule (40).

Completing the Square

The title of this method of solution results from the fact that all terms of the equation involving the variable are transposed to one side of the equal sign after which an appropriate constant term is added so that this side of the equations is arranged to form a perfect square. A *perfect square* may be defined as a quantity which has an exact square root. The other side of the equation consists of the remaining portion of the original equation together with the same constant term as added to complete the square on the side of the equation involving the variables. After getting the equations into the form described, the next operation is to take the square root of both sides simultaneously, and then simplify the resulting equation.

The method of completing the square is used almost altogether in the solution of quadratic equations, although higher equations which can be arranged in the form of perfect squares can be similarly solved.

$$x^2-3x+2=0$$ B23 H3

The equation as given is obviously not in the form of a perfect square. To change it into such form the constant term 2 must be added to or subtracted from according to whether its given value is less or greater than the desired value. The other side of the equation must of course be equally affected, Rule (36). A simpler solution is to transpose the given constant term to the right-hand side of the equation and then determine what value is required for the constant to make the left-hand side a perfect square. Once this is determined, the appropriate number is added and at the same time an equal quantity is added to the other side. Thus the equality of the relationship is unimpaired.

Whenever the coefficient of the x^2 term is equal to one (as in this case), the value of the constant term is necessary to make the left-hand side of the equation a perfect square is equal to the *square* of one-half of the coefficient of the x term. Note that it is the square of one-half of the *coefficient* of the x term and has nothing to do with the variable x.

$$x^2 - 3x = -2$$

$$\left(-\frac{3}{2}\right)^2 = \frac{9}{4}$$

$$x^2 - 3x + \frac{9}{4} = -2 + \frac{9}{4} = -\frac{8}{4} + \frac{9}{4} = \frac{1}{4}$$

$$\left(x - \frac{3}{2}\right)\left(x - \frac{3}{2}\right) \text{ or } \left(x - \frac{3}{2}\right)^2 = \frac{1}{4}$$

These last two equations are equivalent since the square of the factor $(x - 3/2)$ is equal to the left-hand side of the equation above The two right-hand sides of the equation are identical To write the left-hand side of the equation as a factor $(x - 3/2)$ directly from its equivalent expression in the equation above note these facts

1. The first term of the factor is the square root of the first term of the equivalent expression.
2. The second term of the factor is the square root of the last term of the equivalent expression.
3. The sign $\pm$ of the factor is the *same* as the sign of the middle term of the equivalent expression.

$$\left(x - \frac{3}{2}\right)^2 = \frac{1}{4}$$

$$\sqrt{\left(x - \frac{3}{2}\right)^2} = \sqrt{\frac{1}{4}}$$

$$x - \frac{3}{2} = \pm \frac{1}{2}$$

$$x = \frac{3}{2} \pm \frac{1}{2}$$

$$x = \frac{3}{2} + \frac{1}{2} = 2$$

$$x = \frac{3}{2} - \frac{1}{2} = 1$$

Not all quadratic equations encountered will have coefficients of their x^2 terms equal to one, even though the equation is completely simplified. Nevertheless this coefficient can always be made equal to this value by dividing each term of the equation by the coefficient of the x^2 term if its given value is greater than one, or by multiplying by its reciprocal if its value is fractional.

$$4x^2 - 9x + 2 = 0$$

$$x^2 - \frac{9}{4}x = -\frac{2}{4}$$

$$x^2 - \frac{9}{4}x + \left(\frac{9}{8}\right)^2 = -\frac{2}{4} + \left(\frac{9}{8}\right)^2 = -\frac{32}{64} + \frac{81}{64} = \frac{49}{64}$$

$$\left(x - \frac{9}{8}\right)^2 = \frac{49}{64}$$

$$\sqrt{\left(x - \frac{9}{8}\right)^2} = \sqrt{\frac{49}{64}}$$

$$x - \frac{9}{8} = \pm \frac{7}{8}$$

$$x = \pm \frac{7}{8} + \frac{9}{8}$$

$$x = +2;\ x = \frac{1}{4}$$

The operation of changing the equation so that the coefficient of the x^2 term is equal to one is not a necessary operation in all cases. The left hand side of the equation can be arranged in the form of a perfect square by principles explained in the paragraph entitled Factoring, for the arrangement of a polynomial in the form of a perfect square is nothing more than the finding of two identical factors of that polynomial. This procedure is demonstrated in the following example which also shows the convenience and advisability of using the method of completing the square for equations involving irrational roots.

$$5r^2 + 7r - 2 = 0$$

$$100r^2 + 140r = 40$$

$$100r^2 + 140r + 49 = 40 + 49 = 89$$

$$(10r + 7)\ (10r + 7) = 89$$

$$(10r + 7)^2 = 89$$

$$10r + 7 = \sqrt{89} = \pm 9.44$$

$$r = \pm \frac{9.44}{10} - \frac{7}{10} = \pm .944 - .7$$

$$r = .244;\ -1.644$$

Quadratic Formula

Quadratic equations of the form $ax^2 \pm bx \pm c = 0$ and $x^2 \pm bx \pm c = 0$ are so frequently encountered that a formula has been derived to show the value of the variable (x) in terms of the coefficients (a) and (b) and the constant term (c). For simplicity, the general equation $ax^2 + bx + c = 0$ with all of its terms positive is chosen to represent any quadratic equation. From this equation the value of (x) is determined by the method of completing the square. The result of the solution is in the form of an equation, but is more commonly referred to as a formula. By substituting from an equation with numerical values for coefficients and constant terms instead of (a),

(b), and (c), the numerical value of the roots of the equation may readily be determined.

$$ax^2 + bx + c = 0$$

(Dividing equation by a and transposing c)	$x^2 + \frac{bx}{a} = -\frac{c}{a}$
(Adding $\left(\frac{b}{2a}\right)^2$ to both sides of equation)	$x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$
(Factoring left side of equation)	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$
(Taking square root of both sides.)	$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$
(Transposing and simplifying)	$x = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}$

This formula is valid only for equations which are similar to the general equation from which it was derived The specifications for such an equation are as follows:

1 There must be only *one* variable in the equation to be solved. Since (x^2) and (x) are the same variable raised to different powers, there is only one variable present in the general equation as stated above
2 There may be any number of constant terms, (d), (e), (f), (g), and so on, on either side of the equation as assigned, but before solution by the formula all such terms must be transposed and combined into one constant term (c).
3 The sign $(\pm)$ of the terms need not be all positive $(+)$ as in the general equation, $ax + bx + c = 0$, but the sign of the terms must be correctly considered when substituting in the formula.

The formula as derived should be memorized because it is the most widely used method of solution for quadratic equations It is of special convenience, compared to other methods, for solving equations involving either fractional, or irrational roots This is evident by an inspection of the second example.

Since the type of equation solved by this method is of the second degree, there will be two roots from each solution This fact is indicated by the presence of the $\pm$ sign preceding the radical sign $(\sqrt{\ })$ The quantity $b^2 - 4ac$ beneath the radical sign is termed the *discriminate* because its value determines the nature of the roots.

If $b^2 - 4ac > 0$, the roots are real and unequal.
If $b^2 - 4ac = 0$, the roots are real and equal.
If $b^2 - 4ac < 0$, the roots are imaginary

With the possible exception of the last, these three facts seem obvious. If the value of the expression $(b^2 - 4ac)$ is less than zero, that is, it is negative, then the roots are imaginary since the square root of a negative number cannot be represented by real numbers, either positive or negative.

$$5x^2 + 10x + 15 = 30$$
$$5x^2 + 10x - 15 = 0$$

Comparing this with the general form,

$$ax^2 + bx + c = 0,$$

It is obvious that in the equation to be solved

$$a = +5,$$
$$b = +10,$$
$$c = -15.$$

Now, substituting this in the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-10 \pm \sqrt{100 - 4(5)\ (-15)}}{10}$$

$$x = \frac{-10 \pm \sqrt{400}}{10}$$

$$x = \frac{-10 \pm 20}{10}$$

$$x = -3$$

$$x = +1$$

The magnitude of the numbers corresponding to (a), (b), and (c) substituted in the formula for the solution of the above example could have been reduced to smaller integers by dividing the given equation by the coefficient of the (x^2) term. The results obtained in either case are identical because the two equations are equivalent.

$$x^2 + 2x - 3 = 0$$
$$a = 1;\ b = 2;\ c = -3$$

Such simplification as suggested above is advisable in many cases provided that the values for (b) and (c) remain integers. Any fractions resulting from such a division would most likely increase and not decrease the labor involved in the application of the formula.

$$x^2 - 8x + 8 = 0$$
$$a = 1;\ b = -8;\ c = 8$$

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)\ (8)}}{2}$$

$$x = \frac{8 \pm \sqrt{64 - 32}}{2} = \frac{8 \pm \sqrt{32}}{2}$$

$$x = \frac{8 \pm \sqrt{(16)\ (2)}}{2} = \frac{8 \pm 4\sqrt{2}}{2}$$

$$x = 4 \pm 2\sqrt{2}$$

$$x = 4 \pm 2(1.414) = 4 \pm 2.828$$

$$x = 6.828$$

$$x = 1.172$$

If the (a) term or coefficient of (x^2) is not positive as it appears in the equation to be solved, it is advisable to make it positive as a first step in the solution. This is accomplished by multiplying the entire equation by (-1), which is equivalent to changing to the reverse sign each and every term of the equation.

The greatest source of error in solving equations by the use of this method is the failure to correctly apply the correct sign $(\pm)$ for the various terms when substituting in the formula For instance in the formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac},}{2a}$$

the first term to be substituted is $(-b)$ If the number corresponding to (b), which is the coefficient of the (x) term in the equation to be solved, is already negative, then it becomes positive $(+)$ when it is substituted in the equation This is true, since $-(-b) = +b$ The (b^2) term is always positive, since the square of any number, either positive or negative, is always positive In all cases the number corresponding to the (a) term, which is the coefficient of (x^2), will be positive, or can be made positive as already explained In such cases, the expression, $-4ac$, will be minus or plus, respectively, as (c) is plus or minus The term, $(2a)$, in the above case would always be positive

Errors also frequently arise from dividing only part of the numerator by the term $2a$. As shown by the formula, the entire numerator is to be divided, and this is certain to be done if care is taken to extend the divisional bar from the equal sign to the extreme right-hand side of the fraction

The formula method of solution is not limited in its application to quadratic equations

However, any equation solved by use of the formula must be of the same type as a quadratic, that is with the variable of the first term the square of the variable of the second term Examples of this type are shown in the paragraphs entitled Equations of Higher Degree Solved as Quadratics

Radical Equations

Equations in which the unknown quantity appears under a radical sign are called *radical equations* The variable may also appear elsewhere in the equation without the radical sign, and in some cases two or more radical signs are involved in the same equation, each containing the variable

The solution of a radical equation requires that all radical signs be removed This is accomplished by isolating such terms on one side of the equal sign and then squaring both sides of the equation. If the equation contains more than one radical term, transposing and squaring a second or third time may be necessary

This squaring of the terms on both sides of the equal sign raises the degree of the equation. As a result additional roots may be formed which will not check when substituted into the given equation The operation of checking is therefore a necessary part of the solution in this type of problem. Any results which do not check in the original equation are termed extraneous roots and are consequently eliminated

$$
\begin{aligned}
2x+4 &= 6+\sqrt{x+2} \\
2x+4-6 &= \sqrt{x+2} \\
2x-2 &= \sqrt{x+2} \\
(2x-2)^2 &= x+2 \\
4x^2-8x+4 &= x+2 \\
4x^2-9x+2 &= 0 \\
x &= +2 \\
x &= +0.25
\end{aligned}
$$

The original equation is of only the *first* degree, and hence has only *one* root. Since two unidentical values for the root have been found, one of them is obviously false. The correct root may be determined by substituting one of the values in the original equation, and if it satisfies, then it is the true root of the equation. If it does not satisfy, then the remaining value is the correct root. This may be proved by substituting it into the equation. In this example, 2 is found to be the true root of the equation.

If the equation contains more than one radical, it may be necessary to remove one of these radicals at a time by isolating it on one side of the equal sign and then squaring both sides of the equation. A repetition of this process will eventually eliminate all radical terms.

$$
\begin{aligned}
\sqrt{3x-2}-\sqrt{x+3} &= 1 \\
\sqrt{3x-2} &= 1+\sqrt{x+3} \\
3x-2 &= 1+2\sqrt{x+3}+x+3 \qquad \text{Rule (42)} \\
2x-6 &= 2\sqrt{x+3} \\
x-3 &= \sqrt{x+3} \\
x^2-6x+9 &= x+3 \\
x^2-7x+6 &= 0 \\
x &= 6 \quad \text{(true root)} \\
x &= 1 \quad \text{(extraneous root)}
\end{aligned}
$$

Equations of Higher Degree Solved as Quadratics

Many equations of the third or higher degree may be solved without resort to the graphical solution if the given equation can be represented as the product of several factors, or as an equation of lesser degree. Some of the possibilities of such solutions are indicated by the few examples which follow:

$$\frac{144}{y^2}+y^2=25$$

$$
\begin{aligned}
144+y^4 &= 25y^2 \\
y^4-25y^2+144 &= 0 \\
(y^2-16)\,(y^2-9) &= 0 \\
y^2=16 &\qquad y^2=9 \\
y=\pm 4 &\qquad y=\pm 3
\end{aligned}
$$

An alternative solution may be performed by letting $y^2=x$, thus making the given equation a simple quadratic. The formula may then be employed.

$$a = 1; b = -25; c = 144$$

$$y^2 = \frac{25 \pm \sqrt{625 - (4)(144)}}{2} = \frac{25 \pm \sqrt{49}}{2}$$

$$y = \pm 4$$
$$y = \pm 3$$

Equations of the first degree may also be solved as quadratics by the same principle that applies to equations of higher degree as in the previous example.

$$2x + 4 = 6 + \sqrt{x+2}$$
$$2(x+2) - \sqrt{x+2} - 6 = 0$$
$$2(\sqrt{x+2})^2 - \sqrt{x+2} - 6 = 0$$
$$a = 2, b = -1; c = -6$$

$$\sqrt{x+2} = \frac{1 \pm \sqrt{1 - 4(2)(-6)}}{2(2)} = \frac{1 \pm \sqrt{49}}{4} = \frac{1}{4} \mp \frac{7}{4}$$

$$\sqrt{x+2} = \frac{8}{4} = 2 \qquad \sqrt{x+2} = \frac{-6}{4} = -\frac{3}{2}$$

$$x + 2 = 4 \qquad x + 2 = \frac{9}{4}$$

$$x = 2 \qquad x = 1/4$$

These roots should be checked in the original equation. In this example this operation shows $x = 25$ to be an extraneous root. (See Radical Equations).

$$x^6 - 19x^3 - 216 = 0$$

This is really a quadratic in terms of x^3, for if some letter such as (z) is substituted for (x^3), the equation becomes

$$z^2 - 19z - 216 = 0$$

the roots of which are $z = 27$, and $z = -8$

There are in all, six roots for the original equation, but only two are real as shown below.

	$z = x^3 = 27$	$z = x^3 = -8$
	$x^3 - 27 = 0$	$x^3 + 8 = 0$
Factoring:	$(x-3)(x^2+3x+9) = 0$	$(x+2)(x^2-2x+4) = 0$
Roots:	$x - 3 = 0$	
	$x^2 + 3x + 9 = 0$	$(b^2 - 4ac = -27)$
	$x + 2 = 0$	
	$x^2 - 2x + 4 = 0$	$(b^2 - 4ac = -12)$

The discriminates of the two quadratic equations are negative Their roots are imaginary. The only real roots are $x = 3$ and $x = -2$.

In some instances an equation of the third or higher degree can be factored into two or more polynomials, one or more of which is a quadratic. Each of the factors is then solved for the values of the roots This is demonstrated by an example on page 47.

Another example is given below. In the solution it is interesting to note the application of several of the methods of solution which have been described previously.

$$4x^4 - 9x^3 - 21x^2 + 41x - 15$$

By trial and error $x = 1$ and $x = 3$ are found to be roots of the equation. Therefore $(x - 1)$ and $(x - 3)$ are factors of the equation.

$$(x-1)\ (x-3)\ (\qquad) = 0 = 4x^4 - 9x^3 - 21x^2 + 41x - 15$$
$$(x^2 - 4x + 3)\ (\qquad) = 4x^4 - 9x^3 - 21x^2 + 41x - 15$$

$$\begin{array}{r|l}
 & 4x^2 + 7x - 5 \\
\hline
x^2 - 4x + 3 & 4x^4 - 9x^3 - 21x^2 + 41x - 15 \\
 & 4x^4 - 16x^3 + 12x^2 \\
\cline{2-2}
 & \qquad 7x^3 - 33x^2 + 41x \\
 & \qquad 7x^3 - 28x^2 + 21x \\
\cline{2-2}
 & \qquad\qquad -5x^2 + 20x - 15 \\
 & \qquad\qquad -5x^2 + 20x - 15 \\
\cline{2-2}
\end{array}$$

The quotient as found is a prime factor of the given equation. The irrational roots of this quadratic can be found by either completing the square, or by the use of the quadratic formula.

$$\begin{aligned}
(x-1)\ (x-3)\ (4x^2 + 7x - 5) &= 0 \\
4x^2 + 7x - 5 &= 0 \\
x &= 0.54.. \\
x &= -2.3..
\end{aligned}$$

The roots of the given equation are therefore:

$$x = 1;\ x = 3;\ x = 0.54..;\ \text{and } x = -2.3...$$

The same quotient $(4x^4 + 7x - 5)$ could have been obtained by dividing the given expression $(4x^4 - 9x^3 .. etc.)$ by the *binomial* $(x - 1)$ and the quotient thus obtained divided in turn by the binomial $(x - 3)$. The terms binomial and trinomial are used to designate polynomials of two and three terms, respectively.

METHODS OF SOLUTION—SIMULTANEOUS EQUATIONS

The previously outlined methods of solution are applicable to equations containing one variable or unknown, such as $x + 4 = 6$ and $ax^2 \pm bx \pm c = 0$. It must be clearly understood that although this latter equation contains both x and x^2, there is but one unknown in the equation, because x and x^2, are merely different forms of the same unknown, and the finding of one leads directly to the value of the other by a process of substitution. But if the equation contains two or more variables, as in the equation $x^2 + xy + 4 = 3$, then there must be as many independent equations written as there are unknowns appearing in the equations. Each variable is counted *only* once regardless of the number of times it appears. The simultaneous solution of these equations will result in sets of values and not as separate values for x and for y, a fact made clear by the examples included in the paragraphs describing graphical solutions.

When two equations involving two unknowns are solved simultaneously, the num-

ber of pairs of values obtainable can never be greater, and may be less, than the *product* of the degrees of the equations. The degree of any equation is determined by the term with the highest degree. By the degree of a term is meant the sum of the exponents of that term. Thus the terms x^3, xy^2, xyz, are all of the third degree. The equations $x^2 + y^2 = 25$, and $xy = 12$ are both of the second degree.

The simultaneous solution of two or more equations, so that the sets of values obtained will satisfy all of the equations involved can be accomplished by one or more of the several methods now to be described.

Graphical Solutions

The solution of two equations involving two unknowns is given by their point or points of intersection when both equations are plotted to the same scale on coordinate paper. In a single equation of two, and only two, variables, it is impossible to find the value of either one of them unless the value of the other is known. However, if some arbitrary numerical value is assigned to one of these, the equation can be solved for the corresponding value of the other variable. The variable to which the arbitrary value is assigned is called the *independent variable,* and the remaining variable, since it is a function of the independent quantity, becomes the *dependent variable.* In an equation involving (x) and (y), either of the two may be chosen as the independent variable, and the remaining variable consequently becomes dependent. The value of the independent variable is customarily plotted as abscissa and the dependent variable as ordinate. This order may be reversed at any time, even when plotting successive points in the same equation. If the independent variable is represented by (x) and the corresponding value of the dependent variable by (y), a point, P, with coordinates (x) and (y) is established. A number of such points may be similarly plotted and through these points a continuous curve or line may be drawn.

The second equation is similarly plotted and if the curves of the two equations intersect, the abscissa and the ordinate of the points of intersection are the values of (x) and (y) respectively, of these two variables in either equation. The maximum number of points of intersection of two equations is governed by the product of the degrees of the equation as previously explained. This maximum number of intersections is not always realized because of imaginary roots.

An equation of the general form $ax \pm by \pm c = 0$ is called a linear equation since all its plotted points will fall on a straight line. This will always be the case where both of the variables appear separately and with exponents equal to one, but not as their product xy, or their quotient x/y. Since a straight line is determined by knowing two points on it, the graph of a linear equation can be drawn when only two points have been plotted. Usually, but not necessarily always, the most convenient points to choose are those located where the line crosses the two axes. These two points are found by assuming $x = 0$ and finding (y), and then letting $y = 0$ and finding (x). These two values thus found for (x) and (y) are called *intercepts* as the line crosses the axes at these points.

In some cases the X-intercept and the Y-intercept are both zero and consequently the line passes through the origin. This condition is at once apparent whenever the value of the dependent variable is zero for a corresponding zero value of the independent variable. Straight lines which pass through the origin can be plotted whenever one other pair of values of (x) and (y) are known.

(A) $x+y=6$ or $x=6-y$

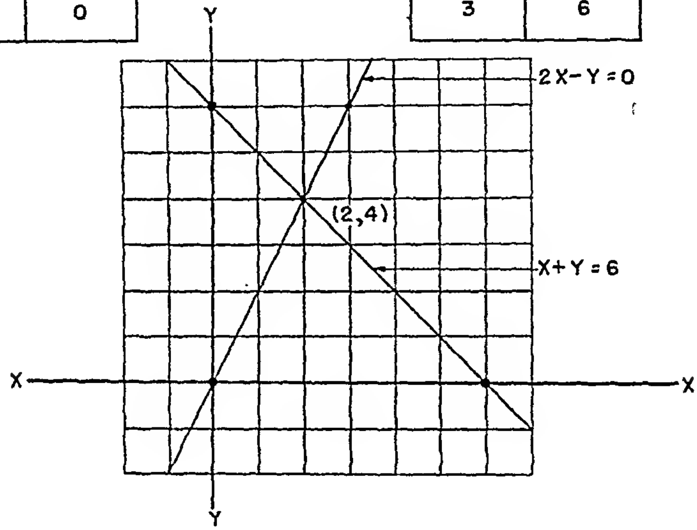
(B) $2x-y=0$ or $x=\frac{1}{2}y$

$X=6-Y$

LET X=	THEN Y=
0	6
6	0

$X=\frac{1}{2}Y$

LET X=	THEN Y=
0	0
3	6



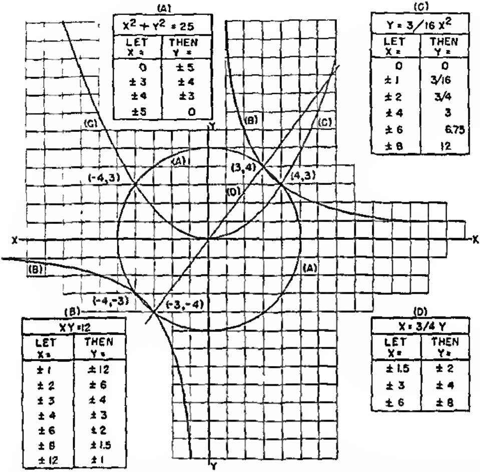
The point of intersection is found to be (2,4) which gives $x=2$ and $y=4$ as the variables which will satisfy both equations, and is therefore the common solution. This fact can be verified by substituting the found values into the given equations. In this case both values of the variable are positive since the point of intersection lies in the first quadrant.

An inspection of the two given equations shows that each of them is of the first degree. The product of the degrees of the equations is therefore one, and the greatest possible set of answers which can be found is one. This fact is apparent for this example without reference to any algebraic theory because two straight lines can intersect at only one point.

Not all equations plot as straight lines, but as curves. In such cases there may be more than one point of intersection, and hence a corresponding number of sets of values found for the variables. The following three examples are all plotted as one diagram to show the general appearance of the four common types of curves, (*A*) circle, (*B*) hyperbola, (*C*) parabola, (*D*) straight line. It should be noted that the maximum number of points of intersections are obtained in the simultaneous solution of equations (*A*) and (*B*) and of equations (*A*) and (*D*). Equations (*A*) and (*C*) apparently have two sets of imaginary roots which of course do not appear on the graph.

(A) $x^2 + y^2 = 25$ (A) $x^2 + y^2 = 25$ (A) $x^2 + y^2 = 25$

(B) $xy = 12$ or $x = \dfrac{12}{y}$ (C) $3x^2 = 16y$ (D) $4x - 3y = 0$



(A)

$X^2 + Y^2 = 25$	
LET X =	THEN Y =
0	± 5
± 3	± 4
± 4	± 3
± 5	0

(C)

$Y = 3/16\ X^2$	
LET X =	THEN Y =
0	0
± 1	3/16
± 2	3/4
± 4	3
± 6	6.75
± 8	12

(B)

XY = 12	
LET X =	THEN Y =
± 1	± 12
± 2	± 6
± 3	± 4
± 4	± 3
± 6	± 2
± 8	± 1.5
± 12	± 1

(D)

X = 3/4 Y	
LET X =	THEN Y =
± 1.5	± 2
± 3	± 4
± 6	± 8

Generally speaking, the graphical solution of two or more equations is not a satisfactory method from the practical viewpoint. Its chief value lies in its ability to picture to the student the meaning of *common solution* and to indicate the significance of *sets of values* for the variables. These facts are important to the understanding of succeeding methods. Additional information concerning straight lines is included in Section V—Analytical Geometry of Straight Lines.

Substitution

Whenever the value of one of the variables can be determined in terms of another variable, the total number of unknowns can be reduced by one. This is especially useful for solving two equations simultaneously when one of the equations includes variables of the first power only. In such cases the first degree equation is solved for the value of one of the variables in terms of the other variable and any accompanying constant terms which may be involved. This computed value is then substituted in the remain-

ing equation in place of the variable to which it is equivalent. The resulting equation involves but one unknown, and is easily solved by previously described methods.

$$\begin{aligned} 2x^2 + 2xy + 10 &= 50 \\ x - y &= 3 \qquad \text{or } x = 3 + y \end{aligned}$$

$$\begin{aligned} 2(3+y)^2 + 2(3+y)y + 10 &= 50 \\ 2(9+6y+y^2) + 6y + 2y^2 + 10 &= 50 \\ 18 + 12y + 2y^2 + 6y + 2y^2 + 10 &= 50 \\ 4y^2 + 18y - 22 &= 0 \\ y &= 1; -5.5 \\ x &= 4; -2.5 \end{aligned}$$

The substitution method can also be used to solve sets of three or more equations. The method consists of solving any one of the equations for (x) or (y) or (z) and substituting the value found in the other equations. For example, in the three simultaneous equations:

$$\begin{aligned} &(A) \qquad 3x + 2y - 4z = 3 \\ &(B) \qquad 2x + y + 3z = 8 \\ &(C) \qquad 5x + 3y + 2z = 14 \end{aligned}$$

Equation (B) is solved for (y) since its coefficient is one. Substituting this value $(y = 8 - 2x - 3z)$ in equations (A) and (C) reduces these equations to:

$$\begin{aligned} x + 10z &= 13 \\ x + 7z &= 10 \\ \hline 3z &= 3 \qquad \text{Subtraction} \\ z &= 1 \\ x &= 3 \qquad \text{By Substitution} \end{aligned}$$

By substituting these values for (x) and (z) in any one of the given equations the value of the remaining variables is found to be $y = -1$. The solution as outlined above may be extended to include sets of equations involving any number of variables. Furthermore the equations may be of any degree and not necessarily linear as used in the example for the sake of simplicity.

Addition or Subtraction

It is often possible to eliminate one of the variables when solving two equations simultaneously by adding together or subtracting one equation from the other, a previous operation having been performed so that the common variables to be eliminated from each equation have equal coefficients. That one equation can be subtracted from another without destroying the equality of the relationship is apparent from Rule (37). In this instance it is not the same but an equal quantity that is being subtracted from both sides of one of the equations.

$$
\begin{array}{lrl}
(A) & 5x^2 + 10y = 85 \\
(B) & 2x^2 - \ \ 2y = 10 \\
 & 5x^2 + 10y = 85 \\
 & 10x^2 - 10y = 50 \\
\hline
(A) + (B) & 15x^2 + \ \ 0 \ = 135
\end{array}
$$

$$x^2 = \frac{135}{15} = 9$$

$$x = \sqrt{9} = \pm 3$$

Certain sets of equations in which (x) and (y) occur in the denominators can be solved by the method of addition or subtraction without first eliminating the fractions.

Example.

$$\frac{5}{x} - \frac{6}{y} = -\frac{4}{3}$$

$$\frac{7}{x} + \frac{4}{y} = \frac{13}{3}$$

$$\frac{10}{x} - \frac{12}{y} = -\frac{8}{3}$$

$$\frac{21}{x} + \frac{12}{y} = \frac{39}{3}$$

$$\frac{31}{x} = \frac{31}{3} \qquad \text{Rule (41)}$$

$$x = 3$$

$$\frac{35}{x} - \frac{42}{y} = -\frac{28}{3}$$

$$\frac{35}{x} + \frac{20}{y} = \frac{65}{3}$$

$$\frac{62}{y} = \frac{93}{3}$$

$$\frac{2}{y} = \frac{3}{3}$$

$$y = 2$$

In the case of a group of three equations to be solved simultaneously, and in each of the equations appear all three of the variables, the solution consists of combining any one of the equations with each of the other two equations. There will be two new equations thus formed, each containing two unknowns These two new equations are now solved simultaneously, and the values of the two unknowns determined. The value of the third unknown is found by substituting the values of the two variables into any one of the original equations.

$$
\begin{array}{ll}
(A) & 2x - y + z = 5 \\
(B) & 3x + 2y + 3z = 7 \\
(C) & 4x - 3y - 5z = -3 \\
\hline
\end{array}
$$

$$
\begin{array}{ll}
(A) & 6x - 3y + 3z = 15 \\
(B) & 3x + 2y + 3z = 7 \\
\hline
(D) & 3x - 5y \qquad = 8
\end{array}
\qquad
\begin{array}{ll}
(B) & 15x + 10y + 15z = 35 \\
(C) & 12x - 9y - 15z = -9 \\
\hline
(E) & 27x + y \qquad = 26
\end{array}
$$

Equations (D) and (E) are now solved simultaneously for (x) and (y), the values of which are substituted in any of the three original equations, and the value of (z) thus determined.

If in a group of three equations to be solved simultaneously, there are but two unknowns appearing in one of the equations, then the other two equations are first solved simultaneously, eliminating the variable not found in the unused equation. The new equation formed will contain but two unknowns, and can be solved simultaneously with the other equation of two variables, and the results used to determine the third variable by a process of substitution in either of the two original equations in which it appears.

$$
\begin{array}{rrl}
(A) & 2x - y + z = 5 & \\
(B) & 3x + 2y + 3z = 7 & \\
(C) & 4x - 3y \qquad = 7 & \\
\hline
(A) & 6x - 3y + 3z = 15 & \\
(B) & 3x + 2y + 3z = 7 & \\
\hline
(A) - (B) & 3x - 5y = 8 & \\
& 3x - 5y = 8 & \\
(C) & 4x - 3y = 7 & \\
\hline
& 12x - 20y = 32 & \\
& 12x - 21y = 21 & \\
\hline
& -11y = 11 & \\
& y = -1 & \\
& x = +1 & \text{(by substitution)} \\
& z = +2 & \text{(by substitution)}
\end{array}
$$

Errors in solving simultaneous equations by the method of addition and subtraction are the result of:

1. The failure to multiply, or divide, each and every term of the given equation by the same factor in the process of making equal the coefficients of one of the variables common to both equations.

2. Improper addition or subtraction. Both positive and negative terms are of common occurrence, and through carelessness or uncertainty in combining the equations, the corresponding terms are often added together in one case and at the same time, subtracted in another. A thorough understanding of the rules applying to positive $(+)$ and negative $(-)$ numbers is essential.

The following example is correctly solved and may be used as a basis of comparison for similar problems of the same type:

$$
\begin{array}{ll}
(A) & .866P + 5T = 100 \\
(B) & \underline{.5P - .866T = 0} \\
(A)\ (5) & .433P + .25T = 50 \\
(B)\ (.866) & \underline{.433P - .75T = 0} \\
(A) - (B) & 1\ 00T = 50 \\
 & T = 50 \\
(A)\ (.866) & .75P + 433T = 86.6 \\
(B)\ (.5) & \underline{25P - .433T = \ \ 0} \\
(A) + (B) & 1\ 00P \qquad = 86\ 6 \\
 & P = 86\ 6
\end{array}
$$

Comparison

Two equations each involving two unknowns as (x) and (y) may be solved by the method of comparison, which is the application of the following rule:

> Solve independently each of the given equations for one of the unknowns—that is, for each equation find (x) in terms of (y) and a constant. Since these two expressions are both equal to (x), they must be equal to each other. This relationship provides an equation involving but one variable (y). Such an equation is readily solved and this numerical value thus found is the value of (y) in either of the two given equations The value of (x) for both equations is found by substituting (y) in either equation and solving for the remaining ing unknown variable

$$
\begin{array}{lll}
(A) & x + y = 15 & \text{or } x = 15 - y \\
(B) & \underline{y - x = \ 1} & \text{or } \underline{x = y - 1} \\
\text{then,} & 15 - 1 = y - 1 & \\
 & 15 + 1 = y + y = 2y & \\
 & y = 8 & \\
 & x = 7 &
\end{array}
$$

Solution of equations by the method of comparison is in reality a solution by the method of Addition and Subtraction previously described. However, it is best to consider it as a separate and distinct solution inasmuch as the procedure, as explained, does not follow the same steps as though it were accomplished by subtracting (B) from (A). The comparison method of solution is especially useful for solving certain types of problems in trigonometry.

Division

The simultaneous solution of two equations by the method of division is an application of Rule (40) In this instance it is not the same but an equal quantity that is used to divide both sides of one of the equations Before this operation is performed it is advisable to make sure that the value of the expression used as a divisor is not equal to zero, as dividing by zero is never a valid operation.

$$
\begin{array}{ll}
(A) & x^2 + xy = 20 \\
(B) & \underline{x + \ y = 10} \\
(A) & x(x + y) = 20 \\
(B) & \underline{(x + y) = 10} \\
(A) \div (B) & x = 2 \\
 & y = 8 \qquad \text{(by substitution)}
\end{array}
$$

An additional example employing division is included in Section III—Trigonometry. This method of solution is especially useful for solving certain types of problems in trigonometry.

RATIO, PROPORTION AND VARIATION

Ratio, Percentage, Proportion

Many problems of everyday occurrence are associated with quantities which are dependent upon some other quantity or quantities to determine their absolute numerical values. Such relationships may be either of a mathematical or a non-mathematical nature depending on the characteristics of the factors involved. Relationships of a non-mathematical nature are often expressed by means of a table or by graphs. If a mathematical relation exists among the various factors involved, the problem can be reduced to a mathematical expression which may take the form of either a proportion or a variation. It is to this type of problem that the following notes apply.

The *ratio* of one quantity to another is the result obtained when the first number is divided by the second. Thus the ratio of 5 to 10 is 5/10 or .5. The ratio of 3 inches to 1 foot (12 inches) is 3/12 or .25. It is important to know that when obtaining the numerical value of any one ratio it is necessary that both quantities be expressed in the same units, that is, dollars or cents, feet or inches, pounds or ounces, etc., but the result obtained will be an abstract number without units (non-dimentional). Ratios are frequently used in everyday life in stating mechanical advantages, efficiencies, specific gravities, etc.

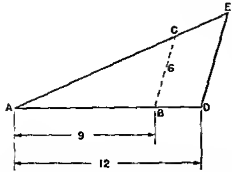
The value of any particular ratio can always be determined by dividing the first term of the ratio by the second term. When the ratio of a part of any quantity to the whole of that quantity is calculated, the result is always a decimal fraction less than unity. For convenience, such ratios are frequently stated as percentage values, or the number of units representing the fractional part of the quantity when the whole of the quantity is assumed to be 100. The value of a ratio can always be expressed as a percentage by multiplying the absolute value of the given ratio by 100.

$$\frac{15}{25} = 0.6 = 60\%$$

Whenever the ratio of two quantities is numerically equal to the ratio of two other quantities, an equation can be written to indicate this relationship. This statement of equality between two ratios is termed a *proportion.*

For example, a theorem of geometry (181) states that if the corresponding sides of two triangles are proportional, the triangles are similar. The converse of this theorem is that if two triangles are similar, the corresponding sides are proportional, that is, the same ratio exists among the corresponding sides. By equating any two of the possible three ratios among the corresponding sides of two similar triangles, a proportion can be established.

In the triangle ADE shown below, the length of side DE is to be determined. Triangle ABC is similar to triangle ADE.

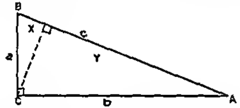


$$\frac{BC}{DE} = \frac{AB}{AD}$$

$$DE = \frac{BC \times AD}{AB}$$

$$DE = \frac{6 \times 12}{9} = 8$$

In this particular example neither of the terms used in one of the ratios of the proportion is used again in the second ratio. This is usually the case, but in the following example an exception to this is shown. Using the same theorem of geometry as before, two proportions may be employed to advantage in developing a proof of the Pythagorean theorem (Rule 186).



The triangle shown above is a right triangle, defined as a triangle which has one of its interior angles a right angle (90°). The side opposite this right angle is termed the hypotenuse

A line drawn perpendicular from the hypotenuse to the vertex (corner) of the right angle divides the original right triangle into two smaller right triangles. The smaller right triangle is similar to the original right triangle since each contains one right angle, and angle B is common to both The third angle of each is also equal since in any triangle the sum of the three interior angles is 180°. Similarly, the second of the smaller right triangles is similar to the original right triangle since each contains one right angle, and angle A is common to both.

Therefore,

$$\frac{x}{a} = \frac{a}{c} \qquad \text{also} \qquad \frac{y}{b} = \frac{b}{c}$$

$$xc = a^2 \qquad \text{and} \qquad yc = b^2$$

$$xc + yc = a^2 + b^2$$ Adding equals to equals, Rule (36)

$$(c)\,(x + y) = a^2 + b^2$$

but $x + y = c$

therefore $c^2 = a^2 + b^2$

The terms (a) and (b) in the proportions,

$$\frac{x}{a} = \frac{a}{c} \quad \text{and} \quad \frac{y}{b} = \frac{b}{c}$$

are specifically designated as the *mean* terms of the proportion, and the remaining two terms in each proportion are designated as the extreme terms. These designations are a result of the older and less desirable way of writing the same proportion which would be stated, (x) is to (a) as (a) is to (c), or algebraically, $x : a : : a : c$. The solution of such a proportion makes use of the fact that the product of the mean terms is equal to the product of the extreme terms. Using the more modern system of denoting ratios as fractions, the first step in the solution is effected by simply finding the cross-product of the terms.

That one of the terms appearing in the first ratio may also be used as one of the terms in the second ratio is apparent by an inspection of the proportions established in the example above. When used in such a manner, the position of the term in the second ratio will be opposite to its position in the first ratio; that is, changed from the numerator to the denominator, or vice versa. Since the same term may be used twice in a single proportion the word "other" as used in the definition of a proportion does not have its usual significance. More specifically, then, a proportion is a statement of equality between *any* two ratios. These ratios may or may not each include a common term.

Another important point which must be understood concerns the dimension of the terms used in writing a proportion. As already stated, each of the two terms of a ratio must be expressed in the same units, but the ratio itself is an abstract number. Therefore in writing a proportion, which is a statement of equality between any two ratios, it is not necessary that the units by which the value of one ratio is determined shall be the same as the units by which the value of the other ratio is determined. An example may be used to demonstrate this point.

If one cubic foot of water weighs 62.4 pounds, what is the weight of 1 gallon (231 cubic inches)?

1 cubic foot = 1728 cubic inches.

Let x = weight of 1 gallon (231 cubic inches).

$$\frac{x \text{ lbs.}}{62.4 \text{lbs.}} = \frac{231 \text{ cu. in.}}{1728 \text{ cu. in.}}$$

$$\therefore (1728)(x) = (231)(62.4)$$
$$x = 8.34 \text{lbs.}$$

If the definitions of ratios and proportions as given are strictly followed there need be no source of error arising from using inconsistent units, inverting one of the ratios, etc. However, the requirements of the definitions can be altered in some instances and the correct result obtained.

In the above example, the solution was obtained by equating the value of one ratio to another. Although such a procedure seems to be the most logical, another method of obtaining the same result is available.

$$\frac{x \text{ lbs.}}{231 \text{ cu. in.}} = \frac{62.4 \text{ lbs.}}{1728 \text{ cu. in.}}$$

$$(1728)(x) = (231)(62.4)$$
$$x = 8.34 \text{ lbs.}$$

Since the cross-products of the two fractions are identical to those obtained by the first method, the solution is correct. However, neither the left nor right side terms of

the equation constitute a ratio, as they do not satisfy the condition that both numerator and denominator be expressed in the same basis It should be noticed, however, that the dimensions of the numerator and denominator of one fraction are consistent, or correspond, to the dimensions of the numerator and denominator, respectively, of the second fraction. This condition must always exist.

The equation as written in the second form is not a proportion, since neither of the terms constituting the expression are ratios. There can, therefore, be no equality of ratios. However, such expressions are frequently called proportions, and since the results obtained are valid, the point of argument seems insignificant in such instances The use of the word ratio is used in another sense not consistent with its mathematical sense For example, the term strength-weight ratios is commonly used to designate the strength of a substance divided by its weight. Such a fraction contains pounds in the numerator and pounds per square inch in the denominator, and is therefore not a true ratio

Direct and Inverse Proportions

The expressions *directly proportional* and *inversely proportional* are frequently employed in everyday conversation When only two variable quantities are being considered, it may be said that they are directly proportional to each other whenever an increase or decrease in one of them produces a proportionate increase or decrease in the other For example, the height of the column of mercury in a thermometer is directly proportional to the temperature.

Two quantities are inversely proportional to each other if an increase in one of them produces a proportionate decrease in the other, or vice versa. For example, the volume of a fixed weight of gas in a cylinder is inversely proportional to the pressure upon it. The word inversely obviously signifies in a reverse order.

The algebraic method of writing a proportion, either direct or inverse, can be demonstrated by using the two examples cited above For the direct proportion:

$$h \propto T \qquad (\propto \text{ means proportional to})$$

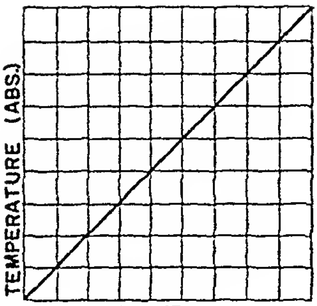
which is read h is proportional to T, the word directly often being omitted when the proportion is of the direct type For the inverse proportion,

$$V \propto 1/P$$

which is read, V is inversely proportional to P.

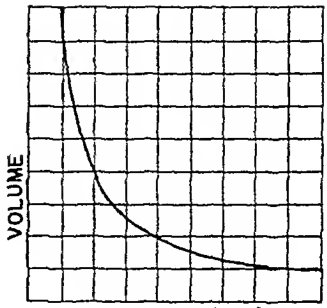
It is at once apparent whether one quantity is directly or inversely proportional to another by observing whether the second quantity appears in the numerator or the denominator of the expression representing the relationship.

Another means of identifying direct proportions and inverse proportions is to observe their graphs. If one quantity is directly proportional to another quantity, a curve representing the relationship will plot as a straight line. If the proportion is an inverse relationship, the curve will plot as a hyperbola. The gas laws known as Charles' and Boyle's laws clearly demonstrate this point.



Charles' Law

$$\frac{V_1}{V_2}=\frac{T_1}{T_2}$$



Boyle's Law

$$\frac{V_1}{V_2}=\frac{P_2}{P_1}$$

Variation

The ideas of *variation* are much the same as in ratio and proportion but in many ways are far more convenient and practical. In mathematical expressions both constants and variables are encountered. As its name suggests, a constant is a number whose magnitude does not change, while a variable quantity may have an unlimited number of values. Constants are represented algebraically by the first letters of the alphabet, (a), (b), (c), etc., or more often by K. Variables are usually represented by (x), (y), (z), etc. Where one quantity (x) varies directly as another quantity (y), the relationship can be written:

$x \propto y$ means proportional to, or varies as.

This is nothing more than a statement of proportionality which may be rewritten as an equation by introduction of a *constant variation.*

$$x = Ky$$
$$\text{or} \quad K = x/y$$

The numerical value of K may be found whenever two instantaneous values of (x) and (y) are obtained, and, once determined, its value remains unchanged throughout the relationship. Fundamentally, variation means nothing more than finding the constant which connects two or more related variables.

The statements, varies directly and varies inversely, have the same significance as direct proportion and inverse proportion. If one quantity varies *directly* as another, an increase or decrease in either one of them will produce a proportionate increase or decrease in the other. If one quantity varies inversely as another, an increase in either one of them will produce a proportionate decrease in the other, and similarly, if one decreases the other one increases. The term inversely has the same significance as indirectly, both terms signifying—in a reverse order. Whether a quantity varies directly or inversely as another is indicated by the position of the second quantity in the equation. In the relation $P = Kb$, the value of P will vary directly as (b), while in equation $P = K/L$, P will vary inversely as L.

It should be observed that if one quantity varies directly as another, the quotient of one quantity divided by the other will always be a constant.

$$P = KL$$

$$\frac{P}{L} = K$$

Also, if one quantity varies inversely as another, the product of the two quantities will always remain constant.

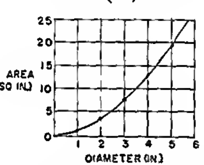
$$P = K/L$$

$$\text{or} \quad PL = K$$

Direct and inverse variations have the same characteristics when plotted as direct and inverse proportions. Because a direct variation always appears as a straight line, such functions are termed straight line or linear variations.

It is often apparent that one quantity varies, either directly or inversely, at some rate other than the first power of the second variable. For instance, the area of a circle varies as the square of either the radius or the diameter.

$$\text{Area} = 3.1416R^2 = 3.1416\left(\frac{D}{2}\right)^2 = .7854D^2$$



In such cases, the equation involving the two variables will not plot as a straight line, but as some type of curve. For example, the equation representing the variation of the area of a circle with the diameter appears as a parabola as shown above. If the exponent of the second variable had been greater than 2, the curve would be steeper (greater slope) and if less than 2, the curve would be flatter (less slope). If the exponential term appears in the denominator, as in the case of an inverse variation, the equation will plot as a hyperbola (See curve B, Page 92).

Joint Variation

In many cases, the magnitude of one quantity is dependent not on one but on several other quantities. Furthermore, the quantity may vary directly as one or more of the quantities, and at the same time vary inversely with respect to the others. The strength of a simple beam can be used to demonstrate this point. Other things being equal, the strength of the beam varies (1) directly as its breadth, (2) directly as the square of its depth, and (3) inversely as its length.

$$S = K_1 b$$

$$S = K_2 (d)^2$$

$$S = \frac{K_3}{L}$$

These three equations may be combined to form a single equation to represent the strength of the beam if the breadth, depth and length are all changed:

$$S = \frac{Kbd^2}{L}$$

This is mathematically correct according to the law of joint variation which states that if a quantity varies directly as two or more quantities, it varies directly as their product; and if a number varies directly as one quantity and inversely as another, it then varies as the quotient of the first divided by the second.

Section II

GEOMETRY

INTRODUCTION

Geometry is a study of the measurement of lines, angles, surfaces, and solids, with their various relations. In plane geometry only figures which lie in one plane are considered. The term plane will not be repeated in the paragraphs which follow, but should be understood to apply in all cases, as only plane geometry is being discussed.

Many of the facts used in geometry are common knowledge which do not require any mathematical proof. These fundamental statements are called axioms and upon them the various propositions of geometry are established. Axioms may be divided into two groups, which are:

> General axioms, or axioms which apply to other kinds of quantities as well as to geometric magnitudes, for instance, to numbers, forces, masses.
>
> Geometric axioms, or axioms which apply to geometric magnitude alone.

The general axioms may be stated as follows:

(48). Things which are equal to the same thing, or to equal things, are equal to each other.
(49). Any quantity may be substituted for its equal in any process.
(50). If equals are added to equals, the sums are equal.
(51). If equals are subtracted from equals, the remainders are equal.
(52). If equals are multiplied by equals, the products are equal.
(53). If equals are divided by equals, the quotients are equal.
(54). Like powers or like roots of equals are equal.
(55). The whole is greater than any of its parts.
(56). The whole is equal to the sum of its parts.

The geometric axioms may be stated as follows:

(57). Through or connecting two given points, only one straight line can be drawn.
(58). A geometric figure may be freely moved in space without any change in form or size.
(59). Through a given point outside a given straight line, one straight line, and only one, can be drawn parallel to the given line.
(60). Geometric figures which can be made to coincide are congruent, that is, they are equal figures.

A geometric postulate is a construction of a geometric figure which, without proof, is admitted as possible.

The geometric postulates may be stated as follows:

(61). Through or connecting any two points, a straight line may be drawn.

(62) A straight line may be extended indefinitely, or may be limited at any point.

(63) A circle may be described about any given point as a center, and with any given radius

Beside the postulates which are used in the actual construction of figures, there are certain other postulates which are used only in the processes of reasoning. As an example, a given angle may be regarded as divided into any convenient number of equal parts Whether it is possible to actually divide this angle on paper by use of the ruler and compass, depends upon practical limitations.

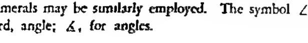
In the paragraphs below, the various terms used in geometry are defined and explained This is followed by a listing of the most important of the geometric propositions

DEFINITIONS

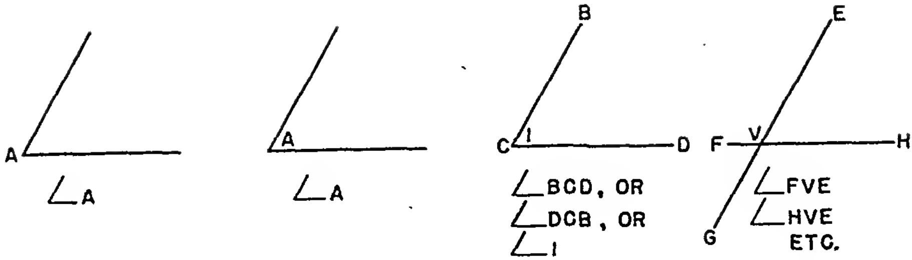
There are many familiar terms which, in geometry, are used with such exactness as to require a precise definition The following terms defined below should be clearly understood and should be consulted from time to time.

(64). A *straight line* is the path of a point moving always in one direction Lines are considered to have length only, not breadth or thickness A straight line is considered as unlimited in its length. The word *line-segment* refers to a line of definite length, or to a part of an unlimited straight line between two of its points A line-segment is identified by two letters, one placed at each end, or by a single letter placed somewhere along its length Arbitrarily chosen capital letters are usually used at the ends of the line, while a single small letter if used is placed along the length of the line

(65) A *curved line* is the path of a point moving in a continually changing direction.

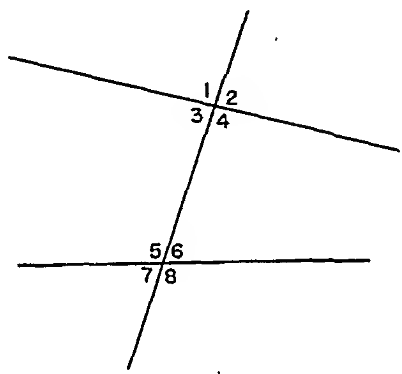


(66). An *angle* is the figure formed by two straight lines drawn from the same point. The point is called the *vertex* of the angle, and the bounding straight lines are called the *sides* (or *legs*) of the angle. An angle is usually identified by an arbitrarily chosen capital letter placed at the vertex, or by capital letters at the vertex and at the ends of the sides. Numerals may be similarly employed. The symbol $\angle$ is used for the word, angle; $\measuredangle$, for angles.



Angles, as used in geometry, are measured in degrees and their subdivisions as explained on page 83.

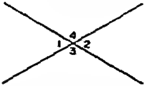
If two straight lines are cut by a third, called a transversal, the angles are named as follows:



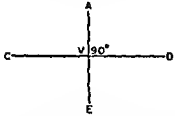
∡ 1, 2, 7, and 8 are *exterior* angles.
∡ 3, 4, 5, and 6 are *interior* angles.
∡ 1 and 8, and 2 and 7, are pairs of *alternate-exterior* angles.
∡ 3 and 6, and 4 and 5, are pairs of *alternate-interior* angles.
∡ 1 and 5, 2 and 6, 3 and 7, and 4 and 8, are pairs of *exterior-interior* angles, often called *corresponding angles.*

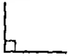
(67). Two angles which have the same vertex and a common side between them are called adjacent angles. Thus, ∡ 1 and 2 are adjacent angles, also ∡ 3 and 4 are adjacent.

(68). Two angles which have the same vertex and the sides of one are the prolongations of the sides of the other are called *vertical angles.* Thus ∡ 1 and 2 are vertical angles, also ∡ 3 and 4 are vertical angles.



(69) If two adjacent angles formed by the intersection of two straight lines are equal, each angle is a *right angle* or 90°. Right angles are indicated hereafter by a small square at the vertex of the angle.





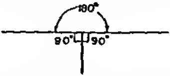
Two straight lines which intersect to form right angles are said to be *perpendicular* to each other, or *normal* to each other. The point of intersection of the perpendicular with the given line is called the *foot of the perpendicular*

(70) An *acute* angle is an angle smaller than a right angle, that is, less than 90°.

(71) An *obtuse* angle is an angle larger than 90°, but not greater than 180°.

(72). Two angles whose sum is one right angle, or 90°, are called *complementary angles*, and either one is said to be the complement of the other. Thus, ∡ 1 and 2 are complementary angles. Angles need not be adjacent to be complementary.

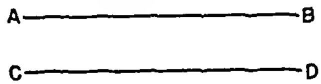
(73) Two sides of an angle which extend in opposite directions from the vertex form a *straight angle*. Such an angle contains two right angles, or 180°.



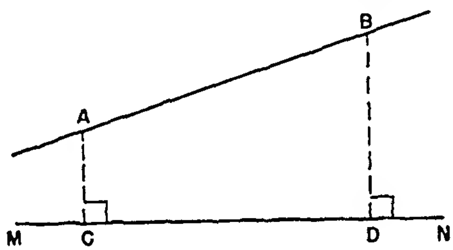
(74). Two angles whose sum is one straight angle, or 180°, are called *supplementary angles*, and either one is said to be the supplement of the other Thus, ∡ 1 and 2 are supplementary angles The angles are not necessarily adjacent.

(75). A *plane surface,* or *plane,* is a surface on which any two points can be connected by a straight line lying wholly in the surface.

(76). Two straight lines lying in the same plane which do not meet no matter how far extended are parallel.



(77). The *projection* of a line segment upon a second line in the same plane is the segment of the second line included between the perpendiculars drawn from it to the extremities of the given line segment. Thus, CD is the projection on line MN of the line segment AB.

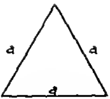


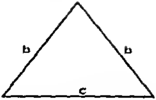
(78). A *polygon* is a plane surface bounded by three or more straight lines. Any polygon contains the same number of angles as it has sides. A *regular polygon* is one that is both equilateral and equiangular, that is, the sides are of equal length and the angles are of equal magnitude. A *diagonal* of a polygon is a straight line jointing any two vertices not adjacent to each other. The *perimeter* of a polygon is the sum of the lengths of its sides.

(79). A *triangle* is a polygon of three sides and consequently three angles. The side connecting any two given angles and common to both angles is called the *included side.* The angle formed by any two given sides of a triangle is called the *included angle.* Triangles which are of exactly the same size and shape are said to be *congruent,* since they can be made to coincide.

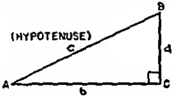
Triangles which are exactly the same shape, but of different size, are called *similar triangles.* The corresponding angles of similar triangles are equal, and the corresponding sides are proportional.

Any triangle that has three equal sides and three equal angles is designated as an *equilateral triangle.* An *isosceles triangle* is one having only two of its sides of equal length.



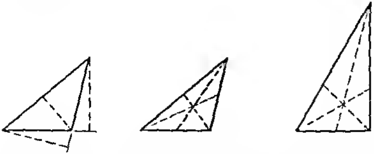


(80). A *right-angled triangle,* or *right triangle,* is one which has one of its interior angles equal to the right angle, or 90° The side of the triangle opposite the right angle is the *hypotenuse.* The length of this side is greater than the length of either one of the other two sides, but is always less than the sum of their lengths The two sides of a right triangle other than the hypotenuse are called *legs.*



(81) An *oblique triangle* is one which does not have one of its interior angles a right angle measuring 90°. Such triangles may or may not contain one angle greater than 90°.

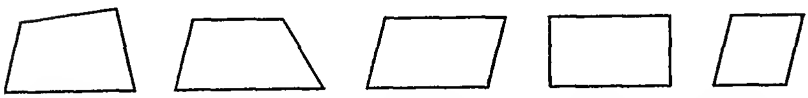
(82) The *altitude* of a triangle is the perpendicular distance from any side (extended, if necessary) to the vertex of the angle opposite that side. A line drawn from the vertex of an angle to the mid-point of the opposite side is called the *median* line A line drawn through the vertex of an angle and dividing the angle into two equal parts is called the *bisector* of the angle



Altitudes Medians Bisectors

(83). A *quadrilateral* is a polygon of four sides. If two of the sides are parallel, the figure is a *trapezoid.* The parallel sides are called *bases,* and the

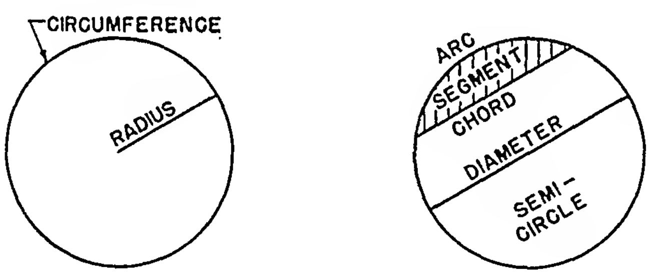
altitude is the perpendicular distance between the two bases. A *parallelogram* is a quadrilateral having both pairs of opposite sides parallel. A diagonal divides a parallelogram into two equal triangles. A *rectangle* is a parallelogram whose angles are all right angles. A parallelogram in which the sides are all of the same length is called a *rhombus.*



Quadrilateral Trapezoid Parallelogram Rectangle Rhombus

(84). Other polygons frequently encountered in geometry have five, six, and eight sides. These are known as *pentagons, hexagons,* and *octagons,* respectively.

(85). A *circle* is a plane figure bounded by a closed curved line called the *circumference,* every part of which is the same distance from a fixed point within, called the *center.* The distance from the center to the circumference is the *radius.* Two circles drawn from the same center, but with different radii, are said to be *concentric.*

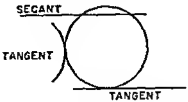


Any part of the circumference is called an *arc,* and a straight line joining the two ends of the arc is called a *chord.* The chord is said to subtend its arc. The area of the circle bounded by an arc and its chord is referred to as a *segment* of the circle.

The circumference of a circle can be divided into any number of arcs. Whenever only two arcs of unequal length are found, the longer arc is called the major arc, and the shorter arc is called the minor arc.

The chord which passes through the center of the circle is the *diameter.* The diameter, therefore, divides the circle into two equal parts called *semicircles.* The length of the diameter is twice the length of the radius. The length of the circumference of a circle divided by the diameter is an irrational number designed by π (Pi), the approximate value of which is 3.1416. The length of the circumference of a circle in terms of the radius or diameter is therefore:

$$\text{Circumference} = \pi D = 2\pi R.$$

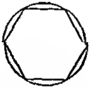


A *secant* to a circle is any straight line intersecting the circle. A *tangent* to a circle is a line which touches the circumference at only one point A tangent line may be either a straight line or a curved line.

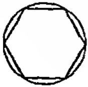
(86) A *central angle* is an angle having its vertex at the center of the circle and radii of the circle for its sides The angle is said to *intercept* the arc included between its sides Thus, ABC intercepts $\widehat{AB}$ (read, arc AB). The area of a circle bounded by the two radii of a central angle and the intercepted arc is called a *sector* of a circle.

An *inscribed angle* is an angle within the circle whose vertex lies on the circumference and whose sides are chords of the circle. An angle is said to be *inscribed* in a segment if the vertex is on the circumference and the sides pass through the ends of the arc of the segment.

(87) A *regular circumscribed polygon* is a regular polygon having all of its sides tangent to a circle The *apothem* of a regular polygon is the radius of its *inscribed circle.*



(88). A *regular inscribed polygon* is a regular polygon having all of its vertices on the circumference of a circle.

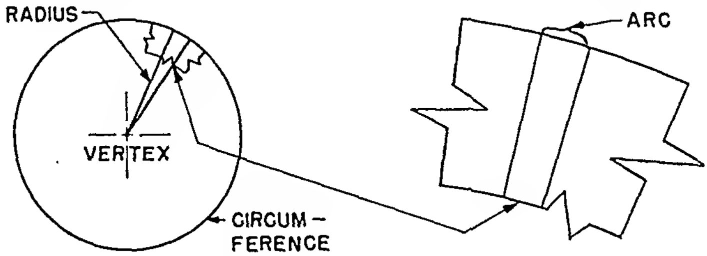


MEASUREMENT OF ANGLES

There are two methods of measuring angles in common use, the sexagesimal method and the circular or natural method. The natural measurement of angles is not essential to the study of geometry, but is here included for completeness.

Sexagesimal Measurement

Sexagesimal measurement of angles is the method most familiar to everyone. It is usually thought of as the Degrees-Minutes-Seconds System as these are the units in which the angles are measured. These units are derived as follows:



A circle is drawn by revolving a line, called a radius, about a fixed point called the vertex or pole. Radial lines are drawn from the vertex so that the circumference is divided into 360 equal arcs. The length of one of these arcs will depend upon the length of the radius; but the angle at the center subtended by one of these divisions will be independent of the radius since it is 1/360 of 360 degrees, a complete circle, or 1 *degree* (°). This unit is divided into sixty parts called *minutes* ('), each of which is subdivided into sixty parts called *seconds* ("). Any angle such as 30 degrees 20 minutes and 15 seconds may be written 30° 20' 15". In many cases it is preferable to define the angle in units of degrees and minutes only. Thus, 30° 20' 15" is also written 30° 20.25' since 15 seconds = = .25 minutes. This latter form is the more convenient for mathematical computations as tables of natural trigonometric functions are usually graduated by degrees and minutes only, with no mention of any such term as seconds. Furthermore, retaining the fraction of a minute as a decimal simplifies to some extent operations as described below.

Regardless of the method of indicating fractional parts of a minute, either as decimal parts of a minute, or as seconds, the operation of adding, subtracting, multiplying or dividing angles measured in the sexagesimal system is somewhat complicated inasmuch as the degrees-minutes-seconds units do not comprise a decimal system. These operation are explained by examples as follows:

When adding angles, place the degrees, minutes, and seconds under each other and add the individual columns. If the operation produces a sum of minutes or seconds greater than sixty, take sixty or a multiple of sixty from that column and for each add one to the column on its left, in order that the answer may be written in its simplest form. Thus to add:

$$\begin{array}{l} 25^\circ\ 30'\ 35'' \\ \underline{35^\circ\ 40'\ 45''} \\ 60^\circ\ 70'\ 80'' = 60^\circ\ 71'\ 20'' = 61^\circ\ 11'\ 20'' \end{array}$$

When subtracting angles, place the degrees, minutes, and seconds under each other and subtract the individual columns. If there are not sufficient minutes or seconds in

the upper number of the column, take one from the column directly to the left of it and add sixty to the insufficient number, as in the following example.

$$\begin{array}{ll} 25^\circ\,45'\,15'' = & 25^\circ\,44'\,75'' \\ \underline{15^\circ\,25'\,35''} & \underline{15^\circ\,25'\,35''} \\ & 10^\circ\,19'\,40'' \end{array}$$

When multiplying angles, multiply each column by the required number, and if the seconds or minutes columns are over sixty, they are reduced the same as in addition. The following example indicates this method

$$\begin{array}{r} 10^\circ\,30'\,40'' \\ \underline{2} \\ 20^\circ\,60'\,80'' = 20^\circ\,61'\,20'' = 21^\circ\,1'\,20'' \end{array}$$

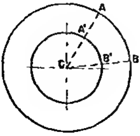
To divide an angle into a given number of parts, divide each column by the number, starting with the degree column The remainder of the dividend is converted into minutes and added to the minutes column, which is then divided by the number. This remainder is then converted into seconds, added to the seconds column, and divided by the number The procedure is shown in the following example, in which an angle 243° 50' 30" is divided by 4

60°	57'	37.5"
4/243	50	30
240	3 × 60 = 180	2 × 60 = 120
3	4/230	4/150
	228	
	2	

It is not always necessary to perform the operation described above These operations are dependent upon the magnitude of the units involved However, to minimize the effort which may be required in such instances, it is advisable to maintain the angles involved in the simplest dimensions possible, that is, in degrees, minutes, and fractions of minutes expressed decimally

Natural Measurement

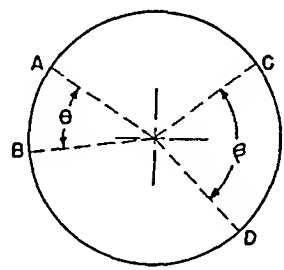
The circular or *natural measurement* of angles is frequently employed in mechanics Instead of dealing with degrees-minutes-seconds, a new unit termed *radian* is employed Do not confuse a radian with a radius A radian is not an arc or a line, but an angle



By geometric reasoning, and from common observation as well, it is known that in any two concentric circles, the arcs intercepted by any angle having its vertex at the center of the circles bear the same relationships to each other as do the radii of the circles. Therefore if ACB is any central angle, then,

$$\frac{\text{arc}\,AB}{AC} = \frac{\text{arc}\,A'B'}{A'C} = constant = k$$

from which it is apparent that the length of the intercepted arc divided by the radius is a number that is always the same for the *same* angle no matter what the radius may be. This is true because the length of the arc at any given angle varies directly as the radius.



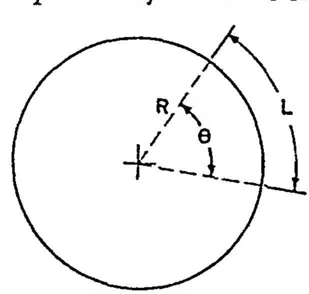
β is the Greek letter Beta.
θ is the Greek letter Theta.

It is also known that in a circle any two central angles are to each other as their intercepted arcs, and consequently as the quotients of their intercepted arcs divided by the radius (since the radius is the same in both cases).

$$\frac{\theta}{\beta} = \frac{\text{arc}\,AB}{\text{arc}\,CD} = \frac{\text{arc}\,AB/R}{\text{arc}\,CD/R}$$

$$\text{or } \theta = \frac{\text{arc}\,AB}{R} \quad \beta = \frac{\text{arc}\,CD}{R}$$

These two quotients as described lead to the definition of circular measurement: The circular measure (in radians) of an angle in a circle is the quotient obtained by dividing the length of its intercepted arc by the radius of the circle.



Thus if θ is the circular measure of the angle (in radians), L its intercepted arc, and R the radius of the circle,

$$\theta = \frac{L}{R}$$

It is apparent that any angle whose measure is one radian will intercept an arc equal in length to the radius, or conversely, an arc equal in length to the radius sub-

tends a central angle of one radian. The magnitude, in radians, of any other angle can be found by dividing the intercepted arc by the radius. Conversely, the length of the intercepted arc is:

$$L = R\theta$$

This relationship is somewhat simpler than if the angle were expressed in degrees If the radius of the circle is assumed to be unity,

$$\theta = \frac{L}{R} = \frac{L}{1} = L$$

And it may, therefore, be said that the circular measure of an angle is represented by the length of the intercepted arc in a circle whose radius is unity.

Conversion of Units

Since the circumference of a circle is equal to $2\pi R$, the central angle (in radians) will be

$$\frac{2\pi R}{R} = 2\pi \text{ radians}$$

In sexagesimal measure a complete circle consists of 360°.

The relation between the two systems of measurement is

$$2\pi \text{ Radians} = 360°$$
$$\pi \text{ Radians} = 180°$$

or $$1 \text{ Radian} = \frac{180°}{3.1416} = 57.3° \quad \text{(Approximately)}$$

or $$1° = \frac{3.1416}{180} = .01745 \text{ Radians.}$$

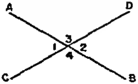
PROPOSITIONS—GEOMETRIC THEOREMS AND PROBLEMS

The term proposition as used in geometry is a general term which includes both geometric theorems and geometric problems A geometric theorem is a statement of truth concerning geometric objects which requires demonstration. A geometric problem is a statement of the construction of a geometric figure, which is required to be made There are also statements called corollaries which are evident truths as a direct consequence of either an axiom or a proposition

The most important of the geometric propositions and corollaries are listed in the following pages. The proof of these statements is not included, but any geometry book will give them completely. It is the propositions themselves, and not their proofs, that are of practical value

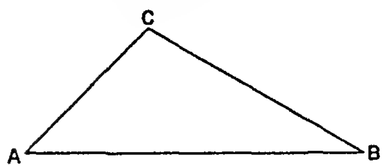
Rectilinear Figures

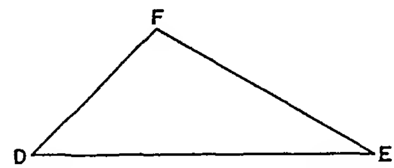
(89). Theorem. If two lines intersect, the vertical angles are equal.



Given the straight lines AB and CD intersecting to form two pairs of vertical angles.
Angle 1 equals angle 2, and angle 3 equals angle 4.

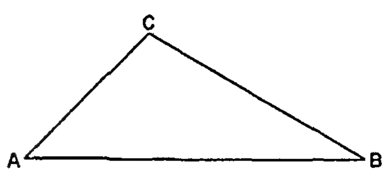
(90). Theorem: Two triangles are congruent if two sides and the included angle of one are equal, respectively, to two sides and the included angle of the other.

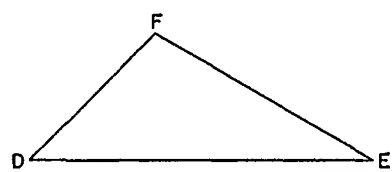




Given the triangles ABC and DEF, with AC equal to DF, AB equal to DE, and angle A equal to angle D.
Triangle ABC is congruent to triangle DEF.
Corollary: Two right triangles are congruent if the legs of the first are equal respectively to the legs of the second.

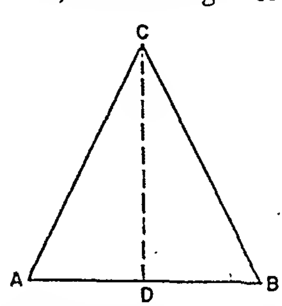
(91). Theorem: Two triangles are congruent if two angles and the included side of one are equal respectively, to two angles and the included side of the other.





Given the triangles ABC and DEF, with angle A equal to angle D, angle B equal to angle E, and AB equal to DE.
Triangle ABC is congruent to triangle DEF.
Corollary: Two right triangles are congruent if a leg and adjacent acute angle of the first are equal respectively to a leg and adjacent acute angle of the second.

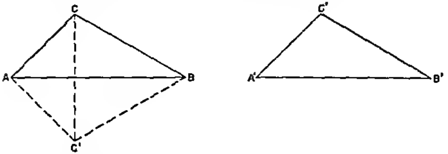
(92). Theorem: If two sides of a triangle are equal, the angles opposite these sides are equal, that is, the base angles of an isosceles triangle are equal.



Given the isosceles triangle ABC in which AC is equal to BC.
Angle A is equal to angle B.

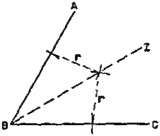
Corollary: The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to the base.
Corollary: An equilateral triangle is also equiangular.

(93). Theorem. Two triangles are congruent if three sides of one are equal respectively to three sides of the other.



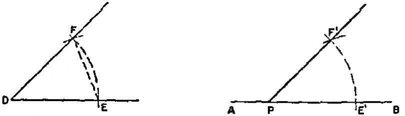
Given the triangles ABC and $A'B'C'$, with AB equal to $A'B'$, AC equal to $A'C'$, and CB equal to $C'B'$.
Triangle ABC is congruent to triangle $A'B'C'$

(94) Problem To bisect a given angle



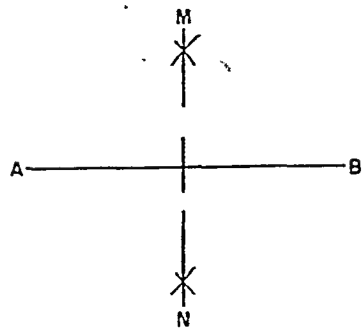
Given angle ABC.
Line BZ bisects angle ABC.

(95) Problem: At a point on a line construct an angle equal to a given angle.



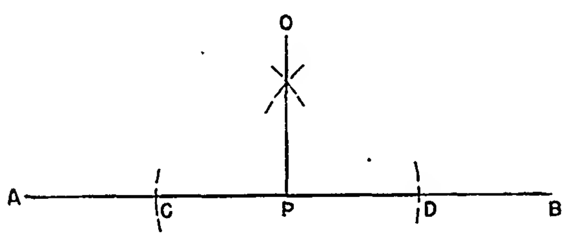
Given line AB and the point P in AB; also, the angle D.
Angle P is equal to angle D.

(96). Problem: To bisect a given line segment; or, to construct the perpendicular bisector of a given line segment.



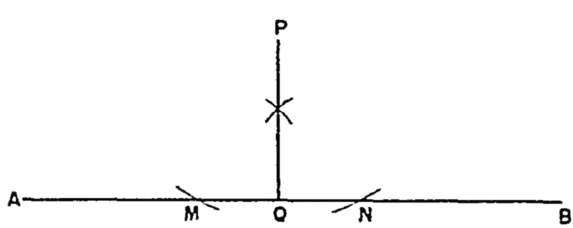
Given the line segment AB.
MN bisects line segment AB.
to the line.

(97). Problem: At a given point in a straight line construct a perpendicular
Given the line AB and the point P in the line.



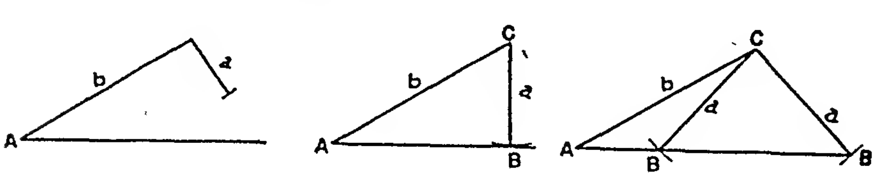
OP is perpendicular to AB at point P.

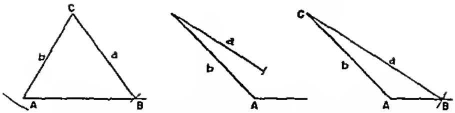
(98). Problem: To draw a perpendicular to a given line from a given external point.



Given the line AB and the point P outside AB.
PQ is perpendicular to AB from point P.

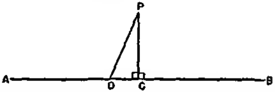
(99). Problem: To construct a triangle, having given two sides and the angle opposite one of them.





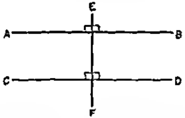
Given (a) and (b) sides of a triangle, and $\angle A$ the angle opposite side (a). Angle A may be either acute or obtuse. The various possible and impossible triangles are shown.
Triangles ABC are formed

(100) Theorem Only one perpendicular can be drawn from a given external point to a given line



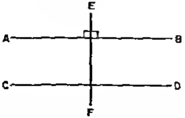
Given the line AB and the external point P. PC is perpendicular to AB, and PD is any other line from P to AB.
PD is not perpendicular to AB

(101). Theorem: Two lines in the same plane perpendicular to the same line are parallel.



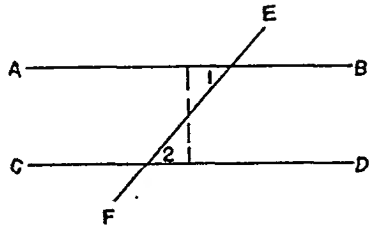
Given two lines AB and CD, in the same plane, both perpendicular to EF.
AB is parallel to CD.

(102). Theorem: If a line is perpendicular to one of two parallel lines, it is perpendicular to the other also.



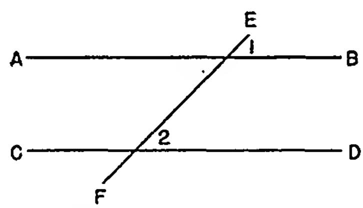
Given AB parallel to CD, and EF perpendicular to AB.
EF is perpendicular to CD.

(103). Theorem: If two parallel lines are cut by a transversal, the alternate interior angles are equal.



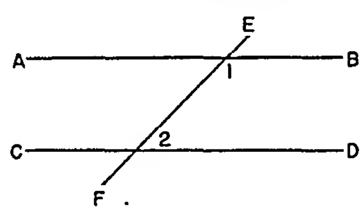
Given two parallel lines *AB* and *CD* cut by the transversal *EF* to form alternate interior angles 1 and 2.
Angle 1 is equal to angle 2.

(104). Theorem: If two parallel lines are cut by a transversal, the corresponding angles are equal.



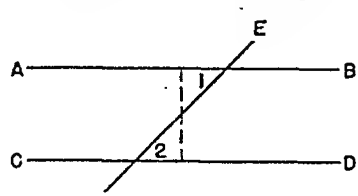
Given two parallel lines *AB* and *CD* cut by the transversal *EF* to form corresponding angles 1 and 2.
Angle 1 is equal to angle 2.

(105). Theorem: If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplementary.



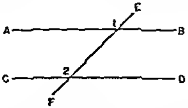
Given two parallel lines *AB* and *CD* cut by the transversal *EF* to form the interior angles on the same side of the transversal 1 and 2.
Angle 2 is supplementary to angle 1.

(106). Theorem: If two lines are cut by a transversal so that a pair of alternate interior angles are equal, the lines are parallel.



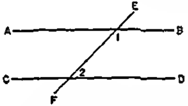
Given two lines *AB* and *CD* cut by the transversal *EF* so that the alternate interior angles 1 and 2 are equal.
AB is parallel to *CD*.

(107). Theorem: If two lines are cut by a transversal so that a pair of corresponding angles are equal, the lines are parallel.



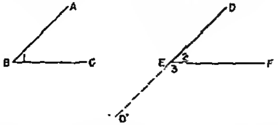
Given two lines *AB* and *CD* cut by the transversal *EF* so that the corresponding angles 1 and 2 are equal.
AB is parallel to *CD*.

(108). Theorem. If two lines are cut by a transversal so that two interior angles on the same side of the transversal are supplementary, the lines are parallel



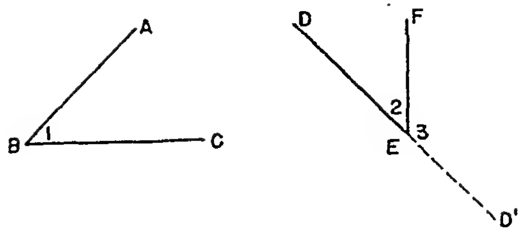
Given two lines *AB* and *CD* cut by the transversal *EF* so that the two interior angles on the same side of the transversal 1 and 2 are supplementary
AB is parallel to *CD*.

(109). Theorem: If the sides of one angle are parallel to the sides of another, the angles are either equal or supplementary.



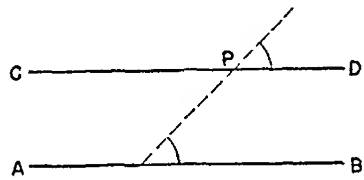
Given the angles *ABC* and *DEF*, with *DED'* parallel to *AB*, and *EF* parallel to *BC*.
Angle 2 is equal to angle 1, and angle 3 is supplementary to angle 1

(110). Theorem: If the sides of one angle are perpendicular to the sides of another, the angles are either equal or supplementary.



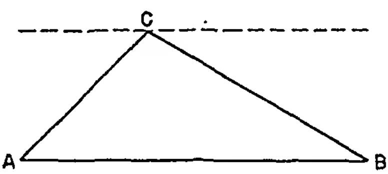
Given the angles *ABC* and *DEF*, with *DED'* perpendicular to *AB*, and *EF* perpendicular to *BC*.
Angle 2 is equal to angle 1, and angle 3 is supplementary to angle 1.

(111). Problem: Through a given point, to draw a straight line parallel to a given straight line.



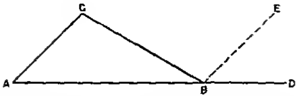
Given the point *P* outside the line *AB*.
CPD is parallel to *AB*.

(112). Theorem: The sum of the three angles of any triangle is equal to 180°.



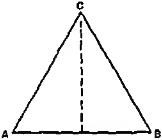
Given the triangle *ABC*.
Angle *A* plus angle *B* plus angle *C* is equal to 180°.
Corollary: The sum of any two angles of a triangle is less than 180°.
Corollary: If one angle of a triangle is a right angle or an obtuse angle, each of the other two angles of the triangle must be acute.
Corollary: In a right triangle, the sum of the two acute angles equals 90°. Each angle of an equilateral triangle contains 60°.
Corollary: If two angles of one triangle are respectively equal to two angles of another triangle, the third angle of the first triangle equals the third angle of the second.
Corollary: If an acute angle of one right triangle equals an acute angle of another right triangle, the remaining acute angles of the triangles are equal.

(113). Theorem: An exterior angle of a triangle is equal to the sum of the opposite interior angles, and is therefore greater than either of them.



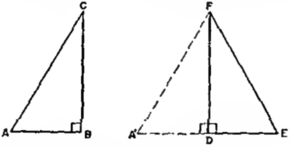
Given the triangle ABC with AB extended to form the exterior angle CBD.
Angle CBD is equal to angle A plus angle C.

(114) Theorem. If two angles of a triangle are equal, the sides opposite these angles are equal.



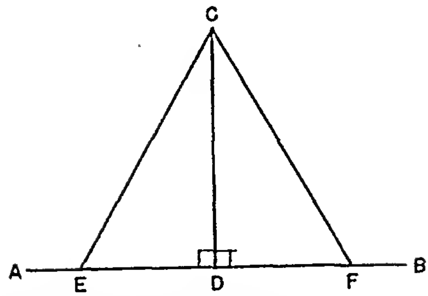
Given the triangle ABC in which angle A is equal to angle B.
AC is equal to BC.
Corollary· An equiangular triangle is also equilateral.

(115) Theorem· If two right triangles have the hypotenuse and a leg of one respectively equal to the hypotenuse and a leg of the other, the triangles are congruent



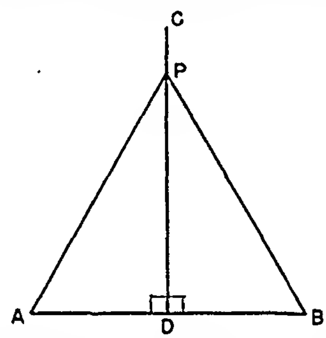
Given hypotenuse AC equal to hypotenuse EF, and leg CB equal to leg FD.
Right triangles ABC and DEF are congruent.
Corollary: Two right triangles are congruent if the hypotenuse and an acute angle of the first are equal respectively to the hypotenuse and an acute angle of the second

(116). Theorem: Two straight lines drawn from a point in a perpendicular to a given line which meet the line at equal distances from the foot of the perpendicular are equal and make equal angles with the perpendicular



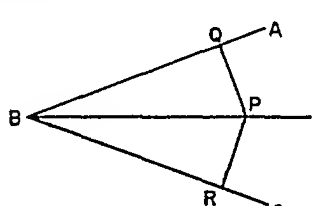
Given: *CD* perpendicular to line *AB*, and oblique lines *CE* and *CF* cutting off equal segments *ED* and *DF*.
CE is equal to *CF*, and angle *ECD* is equal to angle *FCD*.
Corollary: If equal lines are drawn from a point in a perpendicular to a given line, they cut off equal segments on that line from the foot of the perpendicular.

(117). Theorem: Every point in the perpendicular bisector of a line segment is equidistant from the extremities of the line segment.



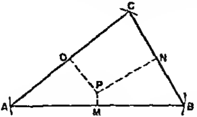
Given *CD* the perpendicular bisector of line *AB*, and *P* any point in *CD*.
PA is equal to *PB*.
Corollary: Conversely, every point equidistant from the extremities of a line lies in the perpendicular bisector of the line.

(118). Theorem: Every point in the bisector of an angle is equidistant from the sides of the angle.



Given *PB* the bisector of the angle *ABC*, *P* any point in *PB*, *PQ* perpendicular to *BA*, and *PR* perpendicular to *BC*.
PQ is equal to *PR*.
Corollary: Conversely, every point equidistant from the sides of an angle lies in the bisector of the angle.

(119) Theorem: The perpendicular bisectors of the sides of a triangle meet in a point equidistant from the vertices of the triangle.



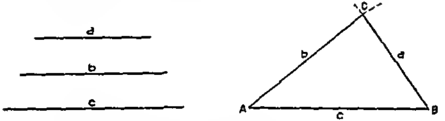
Given the triangle *ABC* with *MP*, *NP*, and *OP* the perpendicular bisectors of the sides

MP, *NP*, and *OP* meet at point *P* which is equidistant from the vertices of triangle *ABC*.

Corollary The bisectors of the angles of a triangle meet in a point which is equidistant from the three sides of the triangle

Corollary The three altitudes of a triangle meet in a point

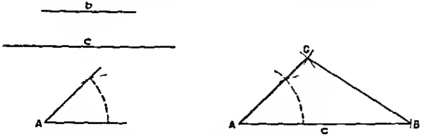
(120) Problem To construct a triangle having its three sides respectively equal to three given straight lines.



Given the lines (*a*), (*b*), and (*c*)

In the constructed triangle *ABC*, *AB* is equal to line (*c*), *BC* is equal to line (*a*), and *AC* is equal to line (*b*).

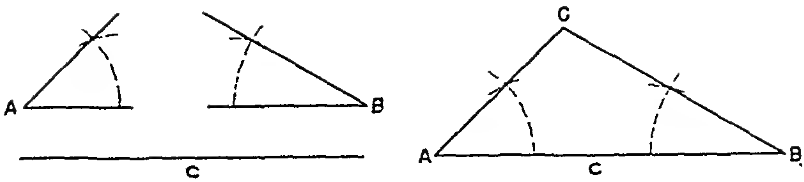
(121) Problem To construct a triangle having two sides and the included angle respectively equal to two given lines and a given angle.



Given the lines (*b*) and (*c*), and the angle *A*.

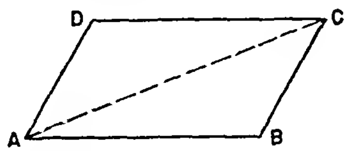
In the constructed triangle *ABC*, *AB* is equal to line (*c*), *AC* is equal to line (*b*), and angle *A* is equal to the given angle *A*

(122). Problem: To construct a triangle having two angles and the included side respectively equal to two given angles and a given line



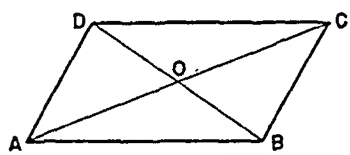
Given the line (c) and the angles A and B.
In the constructed triangle ABC, AB is equal to line (c) and angle A and B are equal to the given angles A and B.

(123). Theorem: The opposite sides of a parallelogram are equal, and the opposite angles are equal.



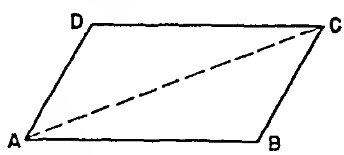
Given the parallelogram $ABCD$.
Side AB is equal to DC and AD is equal to BC, also angle D equals angle B and angle DAB equals angle DCB.
Corollary: A diagonal divides a parallelogram into two congruent triangles.
Corollary: Segments of parallel lines cut off by parallel lines are equal.
Corollary: Two parallel lines are everywhere equidistant.

(124). The diagonals of a parallelogram bisect each other.



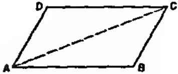
Given parallelogram $ABCD$ with the diagonals AC and BD intersecting at O.
AO equals OC, and DO equals OB.

(125). Theorem: If the two pairs of opposite sides of a quadrilateral are equal, the figure is a parallelogram.



Given the quadrilateral with AB equal to DC and AD equal to BC.
Quadrilateral $ABCD$ is a parallelogram.

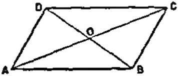
(126). Theorem: If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.



Given the quadrilateral $ABCD$ in which DC is both equal and parallel to AB

Quadrilateral $ABCD$ is a parallelogram.

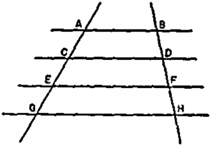
(127). Theorem. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.



Given the quadrilateral $ABCD$ with its diagonals AC and BD intersecting at O, so that AO equals OC and DO equals OB.

Quadrilateral $ABCD$ is a parallelogram.

(128). Theorem If three or more parallel lines intercept equal segments on one transversal, they intercept equal segments on every transversal.

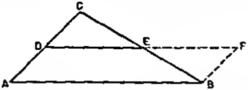


Given the parallel lines AB, CD, EF, and GH intercepting the equal segments AC, CE, and EG on the transversal AG.

Segments BD, DF, and FH are equal.

Corollary If a line bisects one side of a triangle and is parallel to a second side, it bisects the third side also.

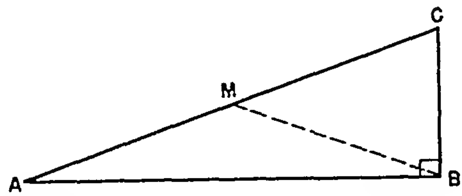
(129). Theorem. The line joining the midpoints of two sides of a triangle is parallel to the third side and equal to one-half the third side.



Given the triangle ABC in which D is the midpoint of AC and E is the midpoint of BC.

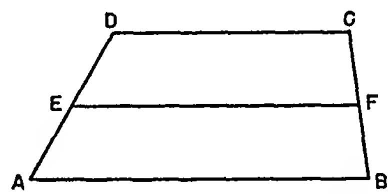
DE is parallel to AB, and DE is equal to one-half AB.

(130). Theorem: The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.



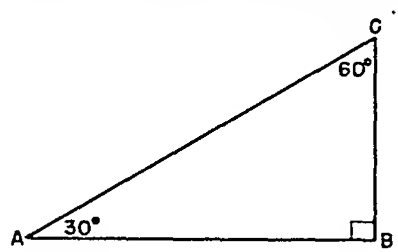
Given right triangle *ABC* with *M* the midpoint of *AC*.
MA, *MB*, and *MC* are equal.

(131). Theorem: The median of a trapezoid is parallel to the bases and equal to one-half their sum.



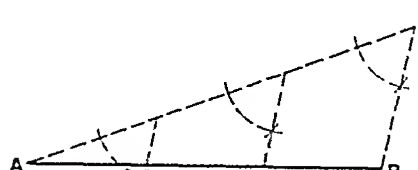
Given *EF* the median of the trapezoid *ABCD*.
Median *EF* is parallel to *AB* and *DC*, also *EF* is equal to one-half the sum of *AB* plus *DC*.

(132). Theorem: In a 30°-60° right triangle, the side opposite the 30° angle is equal to one-half the length of the hypotenuse.



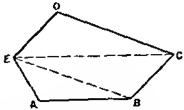
Given right triangle *ABC* with angle *A* equal to 30° and angle *C* equal to 60°.
Side *BC* is equal to one-half the length of *AC*.

(133). Problem: To divide a given line segment into any number of equal parts.



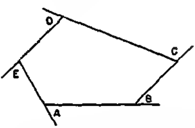
Given the line *AB*.
Line *AB* is divided into three equal parts.

(134). Theorem. The sum of the interior angles of a polygon of N sides is $(N-2)$ times $180°$.



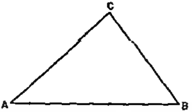
Given polygon $ABCDE$, any polygon having N sides.
The sum of the interior angle A, B, C, D, and E is equal to $(N-2)$ times $180°$.

(135) Theorem The sum of the exterior angles of a polygon, formed by extending the sides in succession, is equal to $360°$.



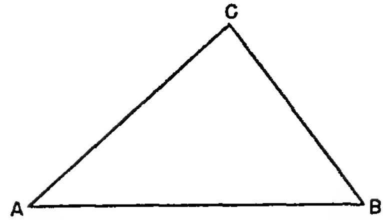
Given the polygon $ABCDE$, any polygon of N sides
The sum of the exterior angles A, B, C, D, and E is equal to $360°$.

(136). Theorem If two sides of a triangle are unequal, the angles opposite these sides are unequal and the angle opposite the greater side is the greater.



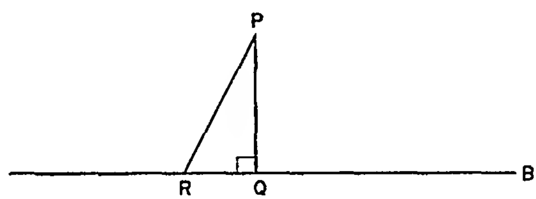
Given triangle ABC with side AC greater than side BC.
Angle B is greater than angle A.

(137). Theorem: If two angles of a triangle are unequal, the sides opposite these angles are unequal and the greater side is opposite the greater angle.



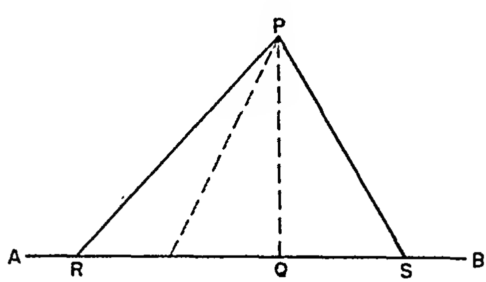
Given the triangle ABC with angle B greater than angle A.
Side AC is greater than side BC.

(138). Theorem: The perpendicular is the shortest line that can be drawn from a given point to a given straight line.



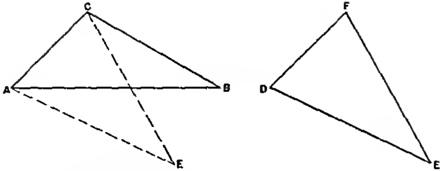
Given the point P and the line AB. PQ is perpendicular to AB, and PR is any other line from P to AB.
Line PQ is less than PR.
Corollary: If a line is the shortest line that can be drawn from a given point to a given line, it is the perpendicular from the point to the line.

(139). Theorem: If two oblique straight lines drawn from a point to a line meet the line at unequal distances from the foot of the perpendicular drawn from the point to the line, the more remote is the greater.



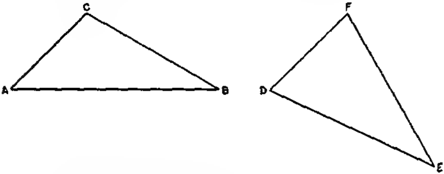
Given the point P outside the line AB. PQ is perpendicular to AB, and QR is greater than QS.
PR is greater than PS.

(140). Theorem: If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first triangle greater than the included angle of the second, then the third side of the first triangle is greater than the third side of the second.



Given the two triangles ABC and DEF with AC equal to DF, CB equal to FE, and angle ACB greater than angle F.
AB is greater than DE.

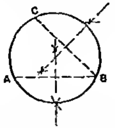
(141). Theorem. If two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, then the angle opposite the third side of the first triangle is greater than the angle opposite the third side of the second triangle.



Given the two triangles ABC and DEF with AC equal to DF, CB equal to FE, and AB greater than DE.
Angle C is greater than angle F.

The Circle

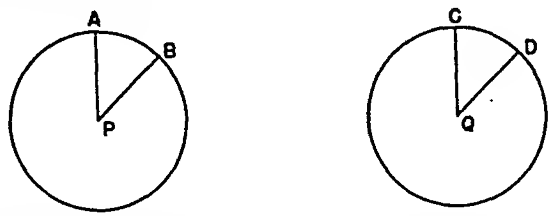
(142). Theorem: Through any three given points not lying in a straight line one circle, and only one, can be drawn



Given points A, B, and C, not in a straight line.
Only the one circle may be drawn through points A, B, and C.

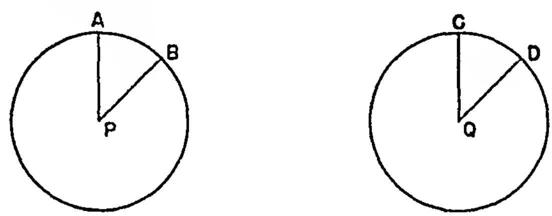
Corollary: The center of a circle circumscribed about a right triangle is the midpoint of the hypotenuse.

(143). Theorem: In the same circle, or in equal circles, equal central angles intercept equal arcs.



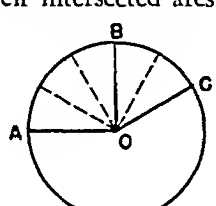
Given the equal circles P and Q with angle APB equal to angle CQD.
Arc AB is equal to arc CD.

(144). Theorem: In the same circle, or in equal circles, equal arcs determine equal central angles.



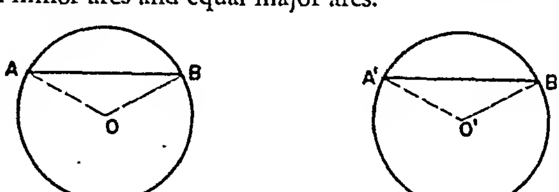
Given the equal circles P and Q with arc AB equal to arc CD.
Angle APB is equal to angle CQD.
Corollary: If in the same circle, or in equal circles, two central angles are unequal, the greater angle intercepts the greater arc; and
Conversely, if two arcs are unequal, the greater arc determines the greater angle at the center.

(145). Theorem: In the same circle, or in equal circles, two central angles have the same ratio as their intersected arcs.



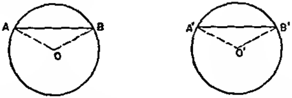
Given central angles AOB and BOC.
Angle AOB divided by angle BOC is equal to arc AB divided by arc BC.

(146). Theorem: In the same circle, or in equal circles, equal chords determine equal minor arcs and equal major arcs.



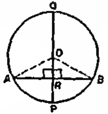
Given the equal circles O and O' with chord AB equal to chord $A'B'$.
Arc AB is equal to arc $A'B'$.

(147). Theorem: In the same circle, or in equal circles, if two arcs are equal, their chords are equal.



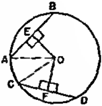
Given the equal circles O and O' with arc AB equal to arc $A'B'$.
Chord AB is equal to chord $A'B'$

(148) Theorem. A diameter perpendicular to a chord of a circle bisects the chord and the arcs determined by the chord.



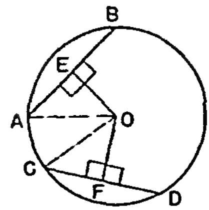
Given the circle O, and the diameter PQ perpendicular to the chord AB at point R
AR is equal to RB, also arc AP is equal to arc PB, and arc AQ is equal to arc BQ.
Corollary A diameter bisecting a chord, which is not a diameter, is perpendicular to the chord
Corollary The perpendicular bisector of a chord passes through the center of the circle.

(149). Theorem In the same circle, or in equal circles, equal chords are equidistant from the center.



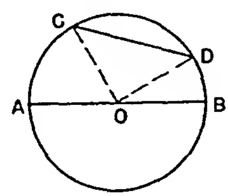
Given the circle O with chord AB equal to chord CD. OE is perpendicular to AB, and OF is perpendicular to CD.
OE is equal to OF.

(150). Theorem: In the same circle, or in equal circles, chords equidistant from the center are equal.



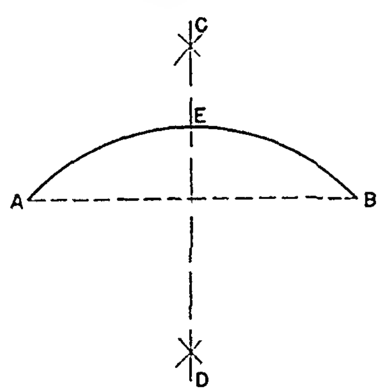
Given the circle O with chords AB and CD. OE is perpendicular to AB, OF is perpendicular to CD, and OE is equal to OF.

(151). Theorem: The diameter of a circle is greater than any other chord.



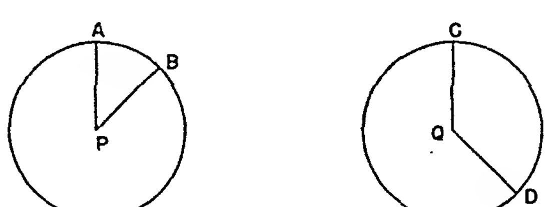
Given the diameter AB and any other chord CD in the circle O. Diameter AB is greater than chord CD.

(152). Problem: To bisect a given circular arc.



Given the circular arc AB.
CD bisects arc AB at point E.

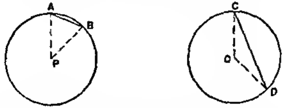
(153). Theorem: In the same circle, or in equal circles, if two central angles are unequal, their arcs are unequal and the greater central angle has the greater arc.



Given the equal circles P and Q with angle CQD greater than angle APB. Arc CD is greater than arc AB.

Corollary: Conversely, in the same circle or in equal circles, if two arcs are unequal, they have unequal central angles, and the greater arc has the greater central angle.

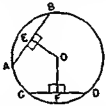
(154). Theorem: In the same circle, or in equal circles, if two chords are unequal, the greater chord determines the greater arc.



Given the equal circles P and Q with chord CD greater than chord AB. Arc CD is greater than arc AB.

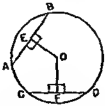
Corollary. Conversely, in the same circle or in equal circles, if two arcs are unequal, the greater arc determines the greater chord.

(155). Theorem: In the same circle, or in equal circles, if two chords are unequal, the shorter is at the greater distance from the center.



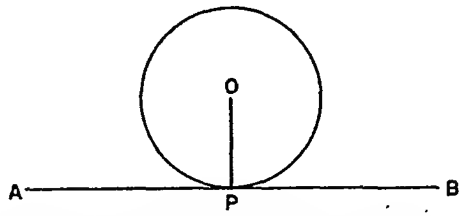
Given the circle O and the chord CD less than chord AB. OE is perpendicular to AB, and OF is perpendicular to CD.
OF is greater than OE.

(156). Theorem. In the same circle, or in equal circles, if two chords are unequally distant from the center, the chord at the greater distance from the center is the shorter.



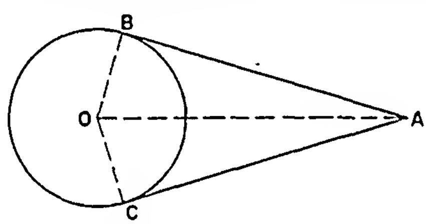
Given the circle O and the chords AB and CD. OE is perpendicular to AB, OF is perpendicular to CD, and OF is greater than OE.
Chord CD is less than chord AB.

(157). Theorem: If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of contact.



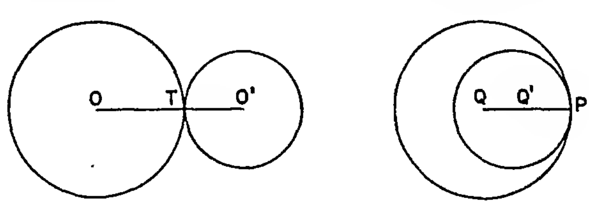
Given AB tangent to the circle O at the point P. OP is a radius.
Tangent AB is perpendicular to the radius OP.
Corollary: A straight line perpendicular to a radius at its outer extremity is tangent to the circle.
Corollary: A perpendicular to a tangent at the point of contact passes through the center of the circle.
Corollary: A perpendicular drawn from the center of a circle to a tangent passes through the point of contact.

(158). Theorem: The tangents to a circle from an external point are equal and make equal angles with the line joining that point to the center.



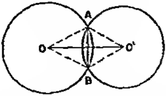
Given the tangents AB and AC from the point A outside the circle O, and AO a line from A to the center O.
Tangent AB is equal to tangent AC, and angle BAO is equal to angle CAO.

(159). Theorem: If two circles are tangent to each other externally or internally, the line of centers passes through the point of tangency.



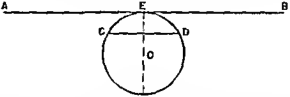
Case I. Given the circles O and O' tangent externally at T.
The line of centers OO' passes through point T.
Case II. Given the circles Q and Q' tangent internally at P.
The line of centers QQ' passes through point P.

(160). Theorem: If two circles intersect, the line of centers is the perpendicular bisector of their common chord.

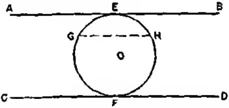


Given the circles O and O', intersecting at A and B, with AB the common chord, and OO' the line of centers.
Line of centers OO' is the perpendicular bisector of the chord AB.

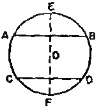
(161). Theorem Parallel lines intercept equal arcs on a circle.
Case I If one line is a tangent and the other is a chord.



Given AB tangent at E to circle O, and parallel to chord CD.
Arc CE is equal to arc DE.
Case II If both lines are tangents

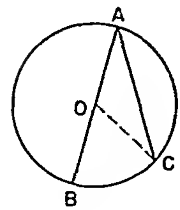


Given the circle O with AB, the tangent at E, parallel to CD, the tangent at F.
Arc EGF is equal to arc EHF.
Case III. If both lines are chords.

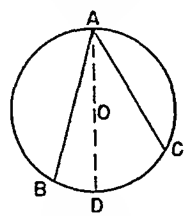


Given the circle O and the chord AB parallel to the chord CD.
Arc AC is equal to arc BD.

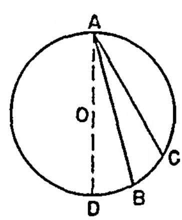
(162). Theorem: An inscribed angle is measured by one-half its intercepted arc.
Case I. If one side of the angle is a diameter.



Given angle BAC inscribed in the circle O, and AB passing through the center O.
Angle BAC is measured by one-half arc BC.
Case II. If the center of the circle lies within the inscribed angle.

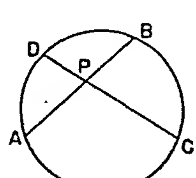


Given the inscribed angle BAC, with the center of the circle O lying within the angle.
Angle BAC is measured by one-half arc BC.
Case III. If the center of the circle lies outside the inscribed angle.



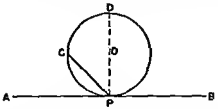
Given the inscribed angle BAC, with the center O outside the angle.
Angle BAC is measured by one-half arc BC.
Corollary: All angles inscribed in the same segment or in equal segments, are equal.
Corollary: An angle inscribed in a semicircle is a right angle, or 90°.
Corollary: An angle inscribed in a segment whose arc is greater than a semicircle is an acute angle. An angle inscribed in a segment whose arc is less than a semicircle is an obtuse angle.

(163). Theorem: An angle formed by two chords intersecting within a circle is measured by half the sum of its intercepted arc and that of its vertical angle.



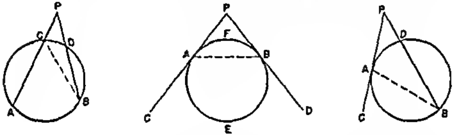
Given chords AB and CD intersecting at the point P.
Angle BPC is measured by one-half the sum of arc BC and arc AD.

(164). Theorem: The angle formed by a tangent and a chord drawn from the point of tangency is measured by half of its intercepted arc.



Given AB a tangent to the circle O at the point P; PC a chord.
Angle APC is measured by one-half arc PC.

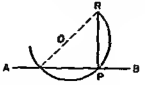
(165). Theorem: An angle formed by two secants of a circle, or by two tangents, or by a secant and a tangent, intersecting at a point outside the circle, is measured by half the difference between the intercepted arcs.



Case I Case II Case III

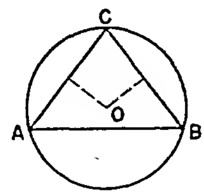
I. Given the circle $ACDB$, and the angle APB formed by the secants PA and PB, meeting at the point P outside the circle, and intersecting the circle at C and D, respectively.
Angle P is measured by half the difference between arc AB and arc CD.
II Given the angle APB formed by the tangents PC and PD touching the circle AEB at points A and B, respectively.
Angle P is measured by half the difference between arc AEB and arc AFB.
III. Given the circle ADB, and the angle CPB formed by the tangent PC touching the circle at point A and the secant PB intersecting the circle at D.
Angle P is measured by half the difference between arc AB and arc AD.

(166). Problem Construct a perpendicular to a given line at a given point.



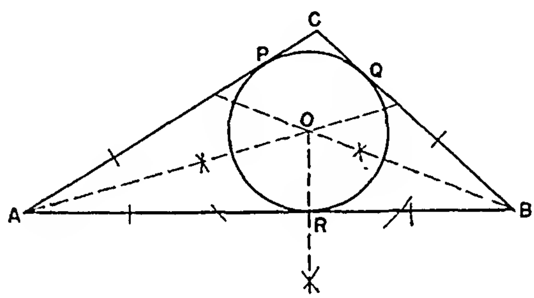
Given the point P in the line AB.
RP is perpendicular to AB at point P.

(167). Problem: To circumscribe a circle about a given triangle



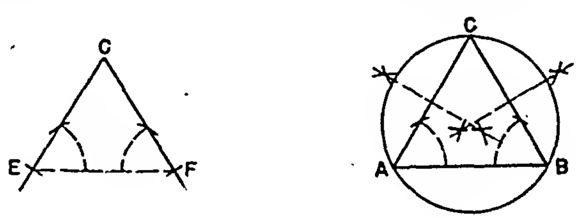
Given the triangle ABC.
The circumscribed circle O passes through the vertices A, B, and C.

(168). Problem: To inscribe a circle in a given triangle.



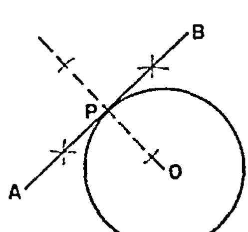
Given the triangle ABC.
The inscribed circle O is tangent at three points, P, Q, and R, to the sides of the triangle ABC.

(169). Problem: On a given straight line as a chord to construct a circular segment in which a given angle may be inscribed.



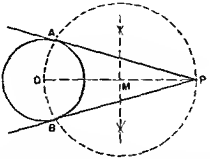
Given the straight line AB and the angle C.
The circular segment ACB constructed on chord AB contains the given angle C.

(170). Problem: To construct a tangent to a given circle at a point on the circle.



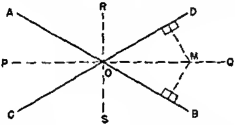
Given the point P on the circle O.
Line APB is tangent to circle O through point P.

(171) Problem. To construct a tangent to a given circle from a given external point



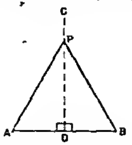
Given P, any point outside the circle O
PA and PB are both tangents to circle O from point P.

(172). Theorem The locus of points equidistant from two given intersecting lines is the pair of lines bisecting the angles formed by the given lines.



Given PQ and RS the bisectors of the angles formed by the intersecting lines AB and CD
The pair of lines PQ and RS is the locus of points equidistant from AB and CD
Corollary The locus of points within an angle equidistant from the sides is the line that bisects the angle

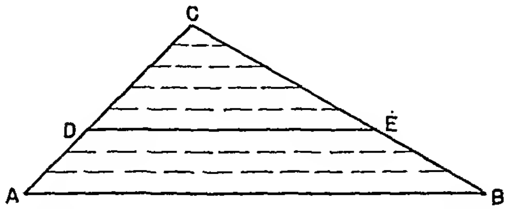
(173) Theorem The locus of points equidistant from two given points is the perpendicular bisector of the line joining them



Given the two points A and B, and CD the perpendicular bisector of AB.
Line CD is the locus of points equidistant from A and B.

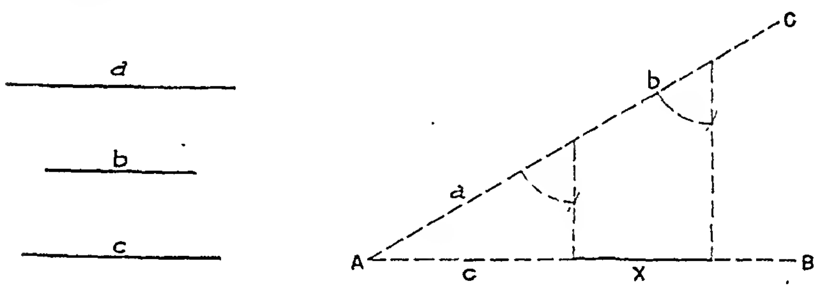
Proportion—Similar Figures

(174). Theorem: A line parallel to one side of a triangle and intersecting the other two sides divides these two sides proportionally.



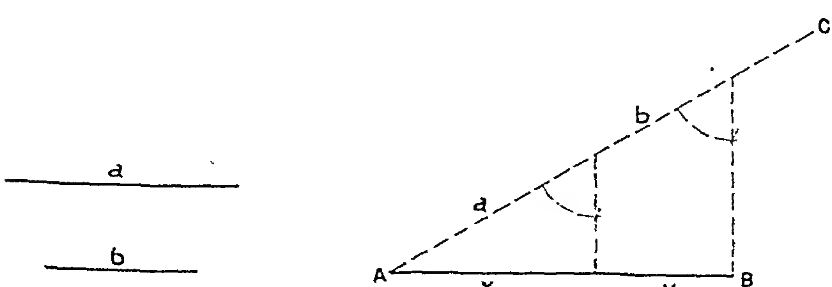
Given the triangle *ABC* with *DE* parallel to *AB* and intersecting the sides *AC* and *CB*.
AD is to *DC* as *BE* is to *EC*.
Corollary: Segments cut off on two transversals by a series of parallel lines are proportional.
Corollary: When two sides of a triangle are cut by a line parallel to the base, one side is to either of its segments as the other side is to its corresponding segment.

(175). Problem: To construct the fourth proportional to three given line-segments.



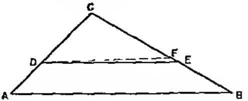
Given the line segments (*a*), (*b*), and (*c*).
Line segment (*a*) is to (*b*) as (*c*) is to (*x*).

(176). Problem: To divide a given line segment into parts proportional to two given line segments.



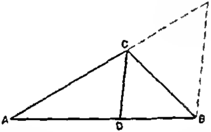
Given the line segment *AB* and the two lines (*a*) and (*b*).
Line *AB* is divided so that (*a*) is to (*b*) as (*x*) is to (*y*).

(177) Theorem. If a line divides two sides of a triangle proportionally, it is parallel to the third side.



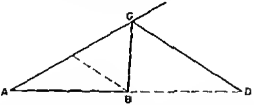
Given the triangle ABC and the line DE intersecting CA and CB so that
CA is to CD as CB is to CE
DE is parallel to AB

(178) Theorem The bisector of an interior angle of a triangle divides the opposite side into segments which are proportional to the adjacent sides



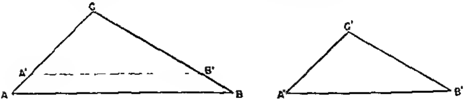
Given the triangle ABC with CD bisecting the angle ACB, and meeting AB at D
AD is to DB as AC is to CB

(179) Theorem The bisector of an exterior angle of a triangle divides the opposite side externally into segments proportional to the adjacent sides



Given the triangle ABC with CD the bisector of the exterior angle at C.
AD is to BO as AC is to BC.

(180). Theorem· If two triangles have the angles of one respectively equal to the angles of the other, the triangles are similar



Given the triangles ABC and $A'B'C'$ in which angle A equals angle A', angle B equals angle B', and angle C equals angle C'.

Triangles ABC and $A'B'C'$ are similar.

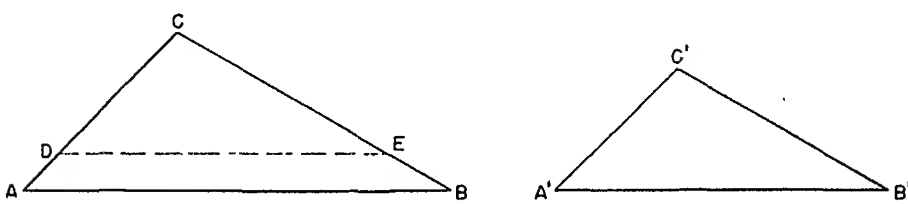
Corollary: If two triangles have two angles of one respectively equal to two angles of the other, the triangles are similar.

Corollary: If two right triangles have an acute angle of one equal to an acute angle of the other, the triangles are similar.

Corollary: If each of two triangles is similar to a third triangle, they are similar to each other.

Corollary: If a line parallel to one side of a triangle cuts the other two sides, a triangle is formed which is similar to the given triangle.

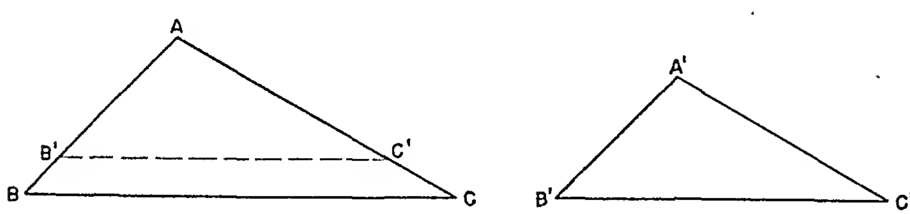
(181). Theoerm: If the corresponding sides of two triangles are proportional, the triangles are similar.



Given the triangles ABC and $A'B'C'$ in which AB is to $A'B'$ as AC is to $A'C'$ as BC is to $B'C'$.

Triangles ABC and $A'B'C'$ are similar.

(182). Theorem: If two triangles have an angle of one equal to an angle of the other, and the sides including these angles proportional, the triangles are similar.

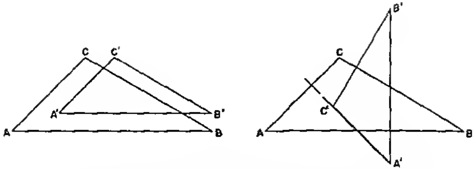


Given the triangles ABC and $A'B'C'$ in which angle A is equal to angle A', and AB is to $A'B'$ as AC is to $A'C'$.

Triangles ABC and $A'B'C'$ are similar.

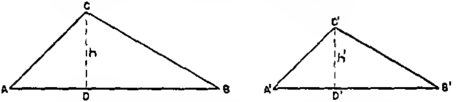
Corollary: Two isosceles triangles in which the vertex angles are equal are similar.

(183). Theorem: Two triangles are similar if their corresponding sides are either parallel or perpendicular to each other.



Given the triangles ABC and $A'B'C'$ in which AB, BC and AC are parallel or perpendicular respectively to $A'B'$, $B'C'$, and $A'C'$.
Triangles ABC and $A'B'C'$ are similar.

(184). Theorem: The corresponding altitudes of two similar triangles have the same ratio as any two corresponding sides.



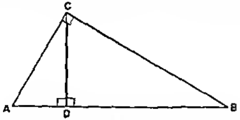
Given the similar triangles ABC and $A'B'C'$ with (h) and (h') corresponding altitudes
Altitude (h) is to (h') as AC is to $A'C'$.

(185). Theorem If, in a right triangle, a perpendicular is drawn from the vertex of the right angle to the hypotenuse:

I The two triangles thus formed are similar to the given triangle and similar to each other

II The perpendicular is the mean proportional between the segments of the hypotenuse

III. Each leg of the given triangle is the mean proportional between the whole hypotenuse and the segment of the hypotenuse adjacent to that leg.



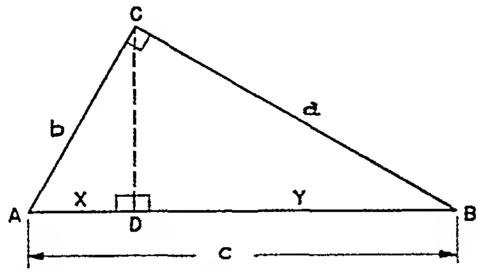
Given the right triangle ACB with the altitude CD cutting off segments AD and DB on hypotenuse AB.

I. Triangles ADC, ACB, and CDB are similar.

II. AD is to CD as CD is to DB.

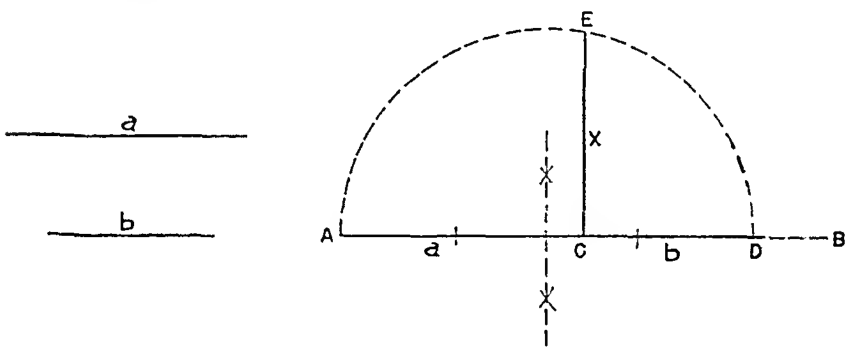
III. AB is to AC as AC is to AD, and AB is to CB as CB is to DB.

(186). Theorem: In a right triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides.



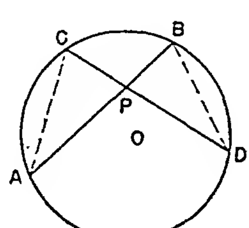
Given the right triangle *ACB* with (c) the hypotenuse and (a) and (b) the legs.
The hypotenuse $(c)^2$ is equal to the sum of $(a)^2$ and $(b)^2$.
Corollary: The square of either leg of a right triangle equals the square of the hypotenuse minus the square of the other leg.

(187). Problem: To construct the mean proportional between two given line segments.



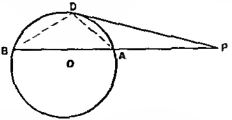
Given the straight lines (a) and (b).
Perpendicular *EC* is the required mean proportional.

(188). Theorem: If two chords intersect within a circle, the product of the segments of one is equal to the product of the segments of the other.



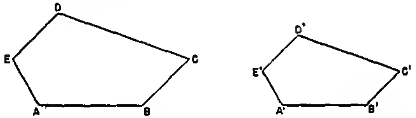
Given the two chords *AB* and *CD* intersecting at *P* within the circle *O*.
Segment *CP* is to *PD* as *AP* is to *PB*.

(189). Theorem. If, from a point outside a circle, a tangent and a secant are drawn, the tangent is the mean proportional between the whole secant and its external segment.



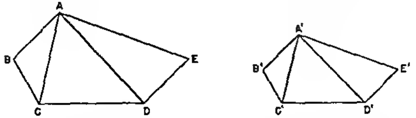
Given the circle O with point P outside the circle, PD a tangent, and PB a secant with AP its external segment.
PB is to PD as PD is to AP

(190) Theorem. The perimeters of two similar polygons have the same ratio as any two corresponding sides.



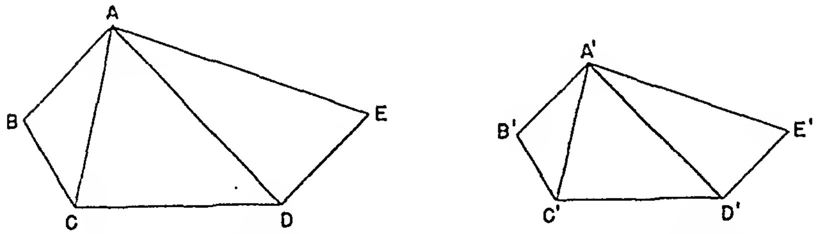
Given the similar polygons $ABCDE$ and $A'B'C'D'E'$ with the perimeters denoted by P and P'.
P is to P' as AB is to $A'B'$.
Corollary The perimeters of two similar polygons have the same ratio as any two corresponding diagonals

(191). Theorem. If two polygons are similar, they can be divided into the same number of triangles, each similar to each other and similarly placed.



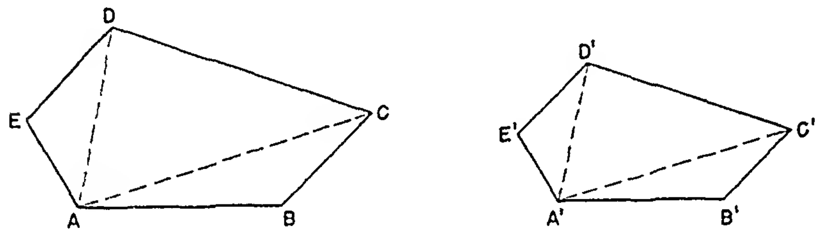
Given the similar polygons $ABCDE$ and $A'B'C'D'E'$ with diagonals drawn from corresponding vertices A and A'.
Each triangle in one polygon is similar to its corresponding triangle in the other polygon.

(192). Theorem: If two polygons are composed of the same number of triangles each similar to each other and similarly placed, the polygons are similar.



Given the polygons $ABCDE$ and $A'B'C'D'E'$ in which the corresponding triangles are similar and similarly placed.
Polygon $ABCDE$ is similar to polygon $A'B'C'D'E'$.

(193). Problem: Upon a given line segment as a side, to construct a polygon similar to a given polygon.



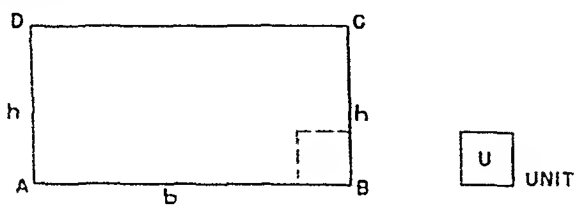
Given a polygon $ABCDE$ and a line $A'B'$ corresponding to the side AB.
Polygon $A'B'C'D'E'$ constructed upon side $A'B'$ is similar to polygon $ABCDE$.

Areas of Polygons

(194). Theorem: The areas of two rectangles having equal altitudes are to each other as their bases.
Corollary: The areas of two rectangles having equal bases are to each other as their altitudes.
Corollary: Two rectangles are equal, if they have equal bases and equal altitudes.

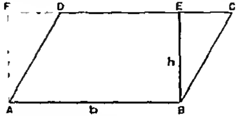
(195). Theorem: The areas of two rectangles are to each other as the products of their bases and their altitudes.

(196). Theorem: The area of a rectangle is equal to the product of its base and its altitude.



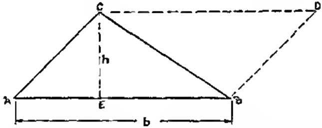
Given the rectangle *ABCD*, with the base containing (b), and the altitude containing (h) units of linear measure.
The area of R is equal to the product of (b) and (h).

(197). Theorem: The area of a parallelogram is equal to the product of its base and its altitude.



Given the parallelogram *ABCD* with the base *AB* equal to (b) and its altitude *BE* equal to (h)
The area of the parallelogram *ABCD* is equal to the product of (b) and (h)
Corollary. Parallelograms with equal bases and equal altitudes are equal in area.
Corollary The areas of two parallelograms are to each other as the products of their bases and their altitudes
Corollary The areas of two parallelograms with equal bases are to each other as their altitudes, and
The areas of two parallelograms with equal altitudes are to each other as their bases.

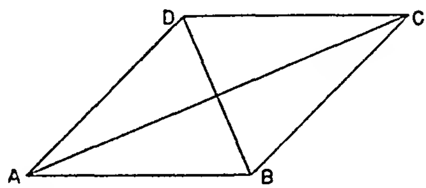
(198) Theorem: The area of a triangle is equal to half the product of its base and its altitude



Given the triangle *ABC* with its altitude *CE* equal to (h) and its base *AB* equal to (b).
The area of triangle *ABC* is equal to one-half the product of (b) and (h).
Corollary: The area of a triangle is equal to one-half that of a parallelogram having the same base and the same altitude.
Corollary: Two triangles which have equal bases and equal altitudes (or equal bases in the same straight line and their vertices in a line parallel to the base) are equal in area.

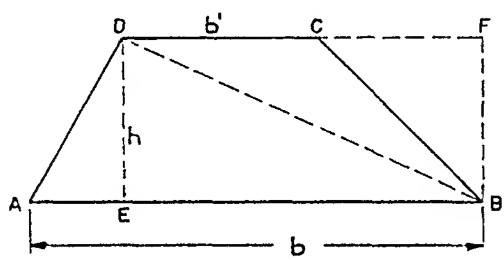
Corollary: Two triangles which have equal bases are to each other as their altitudes; and
Two triangles which have equal altitudes are to each other as their bases.
Corollary: Any two triangles are to each other as the products of their bases and their altitudes.

(199). Theorem: The area of a rhombus is equal to half the product of the diagonals of the rhombus.



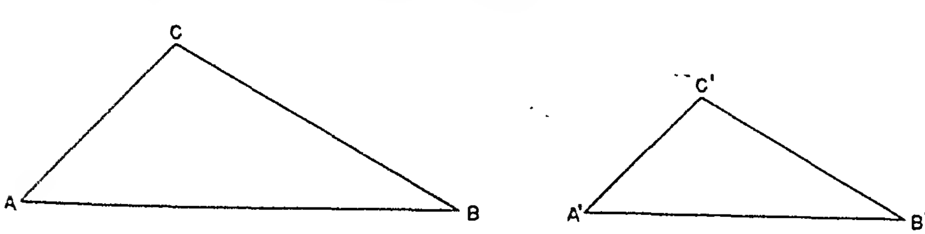
Given the rhombus $ABCD$ with the diagonals AC and DB.
The area of rhombus $ABCD$ is equal to one-half the product of AC and DB.
Corollary: If the diagonals of a quadrilateral are perpendicular to each other, the area of the quadrilateral is equal to half the product of the diagonals.

(200). Theorem: The area of a trapezoid is equal to half the product of its altitude and the sum of its bases.



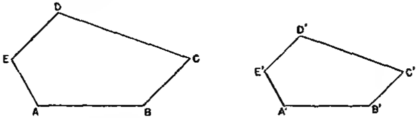
Given the trapezoid $ABCD$ with altitude DE equal to (h), its base AB equal to (b) and its base DC equal to (b').
The area of trapezoid $ABCD$ is equal to one-half the product of (h) times the sum of the bases, (b) and (b').

(201). Theorem: The areas of two similar triangles are to each other as the squares of any two corresponding sides.



Given the similar triangles ABC and $A'B'C'$ with AB and $A'B'$ corresponding sides.
Triangle ABC is to triangle $A'B'C'$ as the square of AB is to the square of $A'B'$.

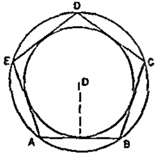
(202). Theorem. The areas of two similar polygons are to each other as the squares of any two corresponding sides.



Given two similar polygons $ABCDE$ and $A'B'C'D'E'$ with AB and $A'B'$ corresponding sides
Polygon $ABCDE$ is to $A'B'C'D'E'$ as the square of side AB is to the square of $A'B'$.
Corollary The areas of two similar polygons are to each other as the squares of their perimeters, or as the squares of any two corresponding lines.

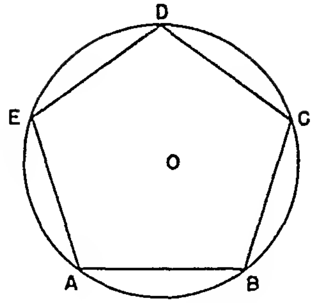
Regular Polygons—Measurement of the Circle

(203) Theorem A circle can be circumscribed about any regular polygon, and a circle can also be inscribed in any regular polygon.



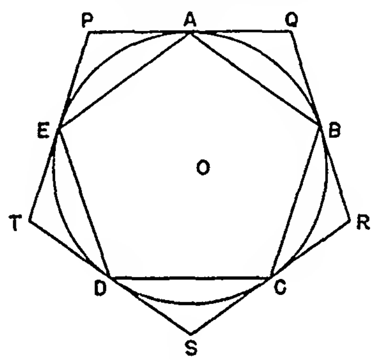
Given the regular polygon $ABCDE$.
The circumscribed circle O passes through the vertices A, B, C, D, E, and the inscribed circle O touches all the sides of the given polygon.

(204). Theorem: An equilateral polygon inscribed in a circle is a regular polygon.



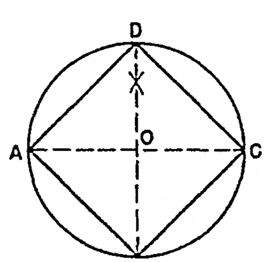
Given the equilateral polygon *ABCDE* inscribed in the circle *O*.
ABCDE is a regular polygon.

(205). Theorem: If a circle is divided into any number of equal arcs,
I. The chords of these arcs form a regular inscribed polygon.



II. The tangents to the circle at the points of division form a regular circumscribed polygon.
Given the circle *O* divided into the equal arcs *AB, BC, CD, DE,* and *EA,* the chords *AB, BC,* etc., and the tangents *PQ, QR,* etc., drawn at the points *A, B,* etc.
ABCDE is a regular inscribed polygon, and *PQRST* is a regular circumscribed polygon.
Corollary: An equiangular polygon circumscribed about a circle is a regular polygon.

(206). Problem: To inscribe a square in a given circle.



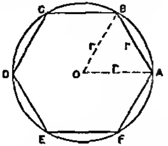
Given the circle *O*.

$ABCD$ is the required inscribed square in circle O.

Corollary: By bisecting the arcs AB, BC, CD, and DA and drawing chords, a regular polygon of eight sides is inscribed.

(207) Problem. To inscribe a regular hexagon in a given circle.

Given the circle O

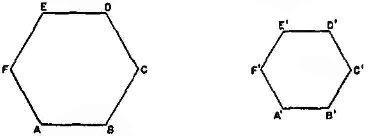


$ABCDEF$ is the regular hexagon inscribed in the circle O.

Corollary: Each side of a regular inscribed hexagon is equal to the radius of the circle.

Corollary. By joining the alternate vertices of a regular inscribed hexagon, an equilateral triangle can be inscribed in a given circle.

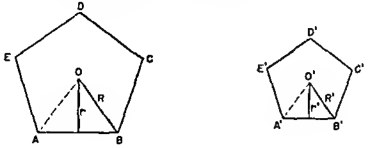
(208). Theorem Regular polygons of the same number of sides are similar.



Given the regular polygons $ABCDEF$ and $A'B'C'D'E'F'$ with the same number of sides

The regular polygons $ABCDEF$ and $A'B'C'D'E'F'$ are similar.

(209). Theorem. The perimeters of regular polygons of the same number of sides are to each other as the radii of the circumscribed circles, or as the radii of the inscribed circles (apothems).

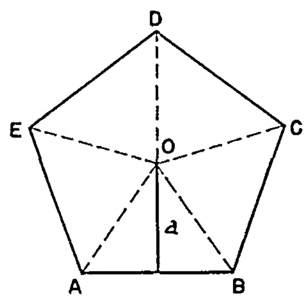


Given two regular polygons of (n) sides, of which P and P' are the perimeters, R and R' the radii of the circumscribed circles, and r and r' the radii of the inscribed circles.

P is to P' as R is to R' as r is to r'.

Corollary: The areas of regular polygons of the same number of sides are to each other as the squares of the radii of the circumscribed, or inscribed, circles.

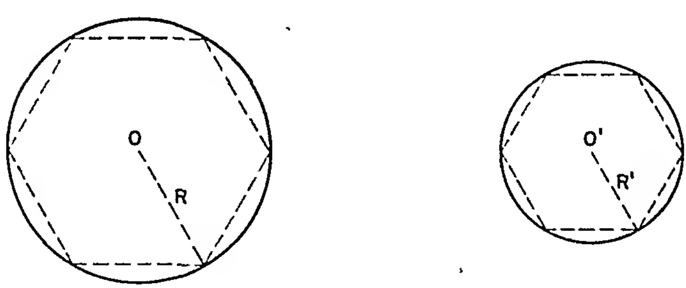
(210). Theorem: The area of a regular polygon is equal to half the product of its apothem and its perimeter.



Given the regular polygon $ABCDE$ with perimeter (p) and apothem (a).

The area of polygon $ABCDE$ is equal to half the product of (a) and (p).

(211). Theorem: The circumferences of two circles are to each other as their radii.

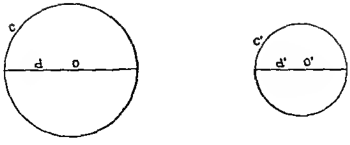


Given the circles O and O', with circumferences C and C' and radii R and R' respectively.

C is to C' as R is to R'.

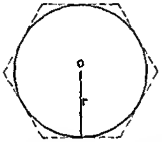
Corollary: The circumferences of two circles are to each other as their diameters.

(212). Theorem: The ratio of the circumference of any circle to its diameter is a constant, symbolized by π, the approximate value of which is 3.1416.



Given the circle O with circumference C and diameter D. Circle O' is any other circle with circumference C' and diameter D'.
C is to D as C' is to D'.

(213). Theorem The area of a circle is equal to half the product of its circumference and its radius



Given the circle O with circumference C, radius R, and area A.
Area A is equal to one-half the product of R and C.
Corollary The area of a circle is equal to the square of its radius multiplied by π
Since, $C = 2\pi R$, $A = \frac{1}{2}(2\pi R)R$, or $A = \pi R^2$.
Corollary The areas of two circles are to each other as the squares of their radii or as the squares of their diameters

$$\frac{A}{A'} = \frac{\pi R^2}{\pi R'^2} = \frac{R^2}{R'^2} = \frac{R'^2}{D'^2}$$

Section III

TRIGONOMETRY

INTRODUCTION

Trigonometry is a branch of mathematics that deals with the measurements of angles and the sides of triangles. This subject involves the use of numbers the same as arithmetic, equations the same as algebra, and also many facts developed in geometry. In *plane* trigonometry these relations are applied to the solution of plane triangles which may be defined as a closed figure of three straight sides all of which lie in one plane. In spherical trigonometry the three sides are not straight lines and do not lie in the same plane. The term plane will not be repeated in the paragraphs which follow but should be understood to apply in all cases as only plane triangles are being considered.

Types of Triangles

The definitions and methods of solution which follow apply to the following types of triangles:

Right triangles, or those triangles in which one of the interior angles is a right angle, that is 90°. The side of the triangle opposite the right angle is the hypotenuse.

Oblique triangles, or those triangles in which none of the interior angles is a right angle. Such triangles may or may not contain one angle greater than 90°.

Elements of Triangles

A triangle consists of six (6) elements—three (3) sides and three (3) angles. If the three angles only (no sides) are given, there will be an infinite number of triangles of the same shape but varying areas which will satisfy the stated conditions. Such triangles are called similar triangles. Consequently, the values of the sides cannot be found. At least one side must be known, and furthermore, the total number of unknown elements must not exceed three. Any triangle, either right or oblique, is said to be completely solved when, having the necessary number of elements given, the remaining elements have been found. In practical work the complete solution is not always necessary as often only the value of one or two of a possible total number of thtee unknown elements is required.

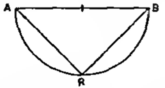
Before proceeding on the solution of any triangle, make certain that the necessary minimum of three elements are known, one of which must be a side. The next fact to be established is whether the given triangle is a right or an oblique triangle, as the latter type are usually solved by special formulas. Right triangles can be solved by the same methods as oblique triangles, but to do so is an unnecessary complication.

Identification of Right Triangles

To determine whether a given triangle is an oblique or a right triangle, apply one or more of the following tests, preferably the algebraic methods number three and four, as number one and two are graphical solutions and therefore subject to slight error.

1 Use a protractor to measure the angle which appears to be a right angle. If its value is 90°, the triangle is a right triangle.

2 Using a compass, construct a circle with its center at the mid-point of the hypotenuse (AB) and with a radius equal to one-half of its length. If the circumference passes through the vertex of the third angle of the triangle, then that angle (R) is a right angle



3. In any triangle the sum of the three interior angles is 180°. At least two of these angles must each be less than 90°. (Such angles measuring less than 90° are termed acute angles, and any angle greater than 90° is called an obtuse angle) If the value of the two acute angles A and B are known, the value of the third angle (R) may be found, by subtracting their sum ($A+B$) from 180°.

$$R = 180° - (A + B)$$

If the remainder is 90°, the triangle is a tight triangle.

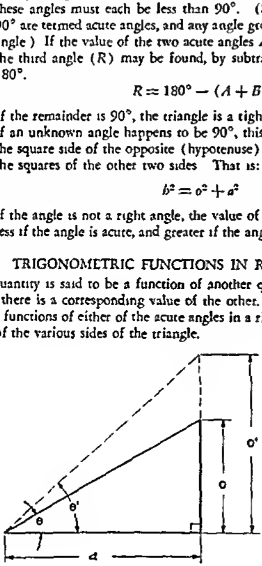
4 If an unknown angle happens to be 90°, this fact will soon be discovered, as the square side of the opposite (hypotenuse) this angle will equal the sum of the squares of the other two sides That is:

$$b^2 = o^2 + a^2$$

If the angle is not a right angle, the value of b^2 will not equal $o^2 + a^2$, being less if the angle is acute, and greater if the angle is obtuse

TRIGONOMETRIC FUNCTIONS IN RIGHT TRIANGLES

One quantity is said to be a function of another quantity if to every value of one quantity there is a corresponding value of the other. As explained below, the trigonometric functions of either of the acute angles in a right triangle are the ratios of the lengths of the various sides of the triangle.

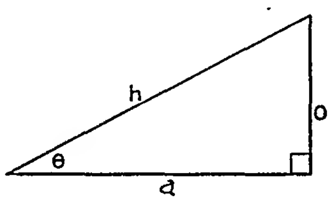


Angle θ is read Theta.

In the triangle drawn in solid lines, o/a is the ratio of the length of the side (o), opposite angle θ, to the length of the side (a), adjacent to angle θ. If the angle at θ is changed to θ', the value of the function o/a becomes o'/a.

Thus it is obvious that the ratio of the two sides (o) *and* (a) of a right triangle depends upon the size of the angle θ. Conversely, the size of the angle θ depends upon the value of the ratio of the sides (o) and (a).

As there are three sides to a triangle, it is possible to have six different ratios of sides. Each one of these ratios is named from its relation to one of the acute angles in a right triangle. Denoting the lengths of the sides of the right triangle by the letters (o) for opposite, (a) for adjacent, and (h) for hypotenuse, the various trigonometric ratios or functions for the angle θ are defined as follows:



Angle β is read Beta.

sine of an angle $= \dfrac{\text{side opposite}}{\text{hypotenuse}}$; written $\sin\theta = \dfrac{o}{h}$; $\sin\beta = \dfrac{a}{h}$

cosine of an angle $= \dfrac{\text{side adjacent}}{\text{hypotenuse}}$; written $\cos\theta = \dfrac{a}{h}$; $\cos\beta = \dfrac{o}{h}$

tangent of an angle $= \dfrac{\text{side opposite}}{\text{side adjacent}}$; written $\tan\theta = \dfrac{o}{a}$; $\tan\beta = \dfrac{a}{o}$

$$= \frac{o}{a} = \frac{o/h}{a/h} = \frac{\sin\theta}{\cos\theta} \qquad \text{(Rule 224)}$$

cotangent of an angle $= \dfrac{\text{side adjacent}}{\text{side opposite}}$; written $\cot\theta = \dfrac{a}{o}$; $\cot\beta = \dfrac{o}{a}$

secant of an angle $= \dfrac{\text{hypotenuse}}{\text{side adjacent}}$; written $\sec\theta = \dfrac{h}{a}$; $\sec\beta = \dfrac{h}{o}$

cosecant of an angle $= \dfrac{\text{hypotenuse}}{\text{side opposite}}$; written $\operatorname{cosec}\theta = \dfrac{h}{o}$; $\operatorname{cosec}\beta = \dfrac{h}{a}$

The word cosine is an abbreviation of the expression, complement of the sine, used centuries ago. An inspection of the definitions given above shows that the sine of any angle is the same ratio, and therefore has the same numerical value, as the cosine of its complementary angle. Similarly, the cosine of any angle has the same numerical value as the sine of its complementary angle. This may be stated algebraically.

$$\sin\theta = \cos(90° - \theta) \qquad \text{(Rule 218)}$$
$$\cos\theta = \sin(90° - \theta) \qquad \text{(Rule 219)}$$

A similar relationship exists between the tangent and the cotangent, and between the secant and the cosecant functions.

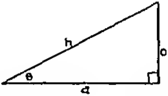
The ratios, contangent, secant and consecant, are not ordinarily necessary as they are merely the reciprocals of the tangent, cosine, and sine respectively. However, these reciprocal functions may be employed to advantage if the solution is being performed by long-hand computations.

GEOMETRIC RELATIONS

The solution of any triangle is accomplished by means of the trigonometric functions and by use of certain geometric relations. The following are the more useful of the geometric relations applying to *right* triangles and also applying to *any* triangle as specified The term any triangle is intended to include both right and oblique triangles Additional geometric relations are contained in SECTION II—GEOMETRY.

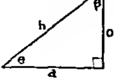
> In a right triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides (Rule 186) This rule is often erroneously applied to oblique triangles.
>
> In a right angle the hypotenuse is greater than either of the two legs, but is less than their sum In any triangle the longest side is always less than the sum of the other two sides
>
> In a righ ttriangle the greater leg is opposite the greater of the two acute angles In any triangle, equal sides are opposite equal engles, and the greatest side is opposite the greatest angle
>
> In a right triangle the sum of the two acute angles must be 90°. If one of the acute angles of a right triangle is equal to one of the acute angles of another right triangle, then the two triangles are similar, since all three of the interior angles are equal. In any triangle, the sum of the three interior angles is 180°.
>
> The corresponding sides of any similar triangles, either right or oblique, are proportional.

The trigonometric ratios and the geometric relations are not independent of each other. For example, if two of the three sides of a right triangle are known, the third side can be found



$$h^2 = a^2 + o^2$$

Also, if one of the trigonometric ratios are given, the other functions can be determined. Thus, if $\tan\theta = 3/4$, and the other ratios are required.



	$\tan\theta = o/a = 3/4$
Let,	$o = 3$
Let,	$a = 4$
Then,	$h^2 = o^2 + a^2 = 9 + 16 = 25$
	$h = 5$
Therefore,	$\sin\theta = o/h = 3/5$ . . . . . also, $= \cos\beta$
Therefore,	$\cos\theta = a/h = 4/5$ . . . . . also, $= \sin\beta$

The angles corresponding to these ratios may be found from a table of natural trigonometric functions as explained below. It must be remembered that the numbers 3, 4, and 5 as used in this particular problem are *relative* values of the sides only. The *absolute* values cannot be found unless the absolute value of one or more of the sides be given. If side (o) has an absolute value of 12, then the absolute value of (h) can be found as follows:

$$o^2 + a^2 = h^2$$

$$\tan\theta = 3/4 = \frac{12}{a}$$

$$\frac{3}{4} = \frac{12}{a}$$

$$a = \frac{(12)\ (4)}{(3)} = 16$$

$$12^2 + 16^2 = h^2$$
$$h^2 = 400$$
$$h = 20$$

A more direct solution makes use of the geometric principle that the corresponding sides of similar triangles are similar.

$$o/h = 3/5 = 12/h$$
$$h = \frac{(5)\ (12)}{(3)} = 20$$

USE OF TABLES OF NATURAL TRIGONOMETRIC FUNCTIONS—INTERPOLATION

The solution of any given right triangle makes necessary the use of a table of the natural trigonometric functions. The word natural is used to differentiate such tables from tables of logarithmic trigonometric tables.

A table of the natural trigonometric functions is a compilation of the numerical values of the more important trigonometric functions, sine, cosine, tangent, cotangent, and in some cases also the secant and consecant, for all angles from 0° to 90°. In most cases the value of these functions are given for each degree and its subdivisions in intervals of one minute. The accuracy of the tabulated values is dependent upon the number of decimal places to which the functions have been calculated, and for ordinary purposes any table of four, five, or six places is considered to be sufficiently accurate.

The explanations and solutions of examples throughout Section III are based on data obtained from a five place table of natural trigonometric functions. However, the same procedure is employed if tables of either less or greater accuracy are employed.

If the given angle is measured in degrees only, or degrees and minutes only, the value of any one of the trigonometric functions can be read directly from the tables. If the given angle is measured in degrees, minutes, and seconds or degrees, minutes, and fractions of a minute, the exact value of the desired trigonometric function cannot be read directly from the table. In many instances the value of the function for the angle in the table which is closest to the given angle is used, the next smaller angle being used if the seconds of the given angle is less than 30, and the next larger angle used if the seconds exceed 30.

The reverse process of finding an angle corresponding to a given or computed value of one of the trigonometric functions, when this value of the function lies in between two of those given in the tables, is similarly accomplished In this case the given function is assumed to be the same as the value of the adjacent value to which it is numerically closest Where more accurate answers are required than is possible by the procedure described above, the following additional computations are employed.

The operation of finding the value of a variable quantity between those given in a table is called interpolation One way to interpolate is to plot a smooth graph of the tabulated values and read off the required value corresponding to the given number. Another method, and of more practical importance, is by the use of proportional parts, or by proportion. Using this solution in connection with the trigonometric functions it is necessary to assume that the change in the value of the function (sine, cosine, and tangent, etc) is directly proportional to the change in the value of the angle. This relationship actually does not exist; however, interpolation carried out in this manner will in general cause no error of any consequence

For those who are mathematically inclined the following facts in regard to interpolation are included, but to those not interested, it is recommended that the material of this paragraph be omitted and their attention be directed to the paragraph beyond.

The operation of finding the value of any trigonometric function between two adjacent angles given in the table is based on the assumption that the change in the value of the function is directly proportional to the change in the value of the angle. This assumption is not true for the table as a whole inasmuch as it would require, for example, that the sine of 60° should be twice the sine of 30°. As an additional proof of the inaccuracy of the statement, the rate of change of the sine is .00029 for one minute in the small angle range and this rate decreases to approximately zero for angles approaching 90°. The rate of change in the value of the tangent function is even more irregular since it increases at an infinite rate for an angle just less than 90°. However, if it is remembered that the values of the various functions are graduated for each degree and minute or 3600 values for an angular change of 90°, it is apparent that the total change in function for one division of the table is very small This is especially true for the functions sine and cosine which have a total variation in value of only unity throughout a range of 90°. The value of the tangent function varies from 0 to unity at 45° and above 45° the function increases more rapidly especially with the larger angles which have values of the tangent function approaching infinity as the angle approaches 90°. Since the tabular difference for the tangent function is greater than that for either the sine or cosine function, a small error in the tangent of an angle will affect the angle less than would a corresponding error in either the sine or the cosine. Consequently, an angle should be determined by means of the tangent function wherever practicable.

Also, when the angle is less than 45°, the tabular difference for the sine exceeds that of the cosine, and when the angle is greater than 45°, the tabular difference for the cosine is the greater. Therefore, the angle should be determined by means of its sine rather than its cosine when the angle is less than 45°, and by its cosine rather than its sine when it is greater than 45°. This difference in accuracy among the tangent, sine, and cosine functions is not very great however, and too much emphasis should not be assigned to this choice of functions. Special formulas for precise interpolation have been developed which may be employed when maximum possible accuracy is required.

Interpolation is frequently required in solving problems involving the six trigonometric functions,

sine—cosine—tangent—cotangent—cosecant—secant

The values of the last two of these are not ordinarily included in the tables as they are the reciptocals of the sine and cosine, respectively. The method of finding the value of the tangent is similar to the method of finding the sine of an angle, and the method for finding the cotangent is similar to the method of finding the cosine of an angle. It is therefore necessary to describe only the operations of finding the sine and the cosine of an angle since the same procedure is employed in interpolating for values of the other functions.

Sine of an Angle

The sine of an angle which is given in degrees, minutes and seconds is found by first finding the value of the function corresponding to the major part of the angle as defined in terms of degrees and minutes only. To this value of the function is *added* an additional number resulting from multiplying (A), the tabular difference between the sines of the two adjacent angles which appear in the tables, one angle being larger than the given angle, and the other angle being smaller than the given angle, by (B), the value of the fraction obtained when the seconds of the given angle are expressed as a fraction of a minute, either as a common fraction as number of seconds/60, or as a decimal.

In many cases the measure of the angle will be given directly in degrees and minutes and decimal parts of a minute thus facilitating the computations being described.

Note that when obtaining the sine of an angle, the value corresponding to the fraction part of a minute or seconds of the angle is always *added* to the value of the function as obtained directly from the table. This is because the value of the sine increases with incteasing magnitude of the angle.

18°

	′	Sine
sine	15	.31316
18°	16	.31344
15.25		

sine 18° 15.25′ =
sine 18° 15′ = .31316
sine 18° 16′ = .31344
Tabular difference = .00028
(.25) (.00028) = .00007
sine 18° 15.25′ = .31316 + .00007
= .31323

Considerable effort can be spared if the tabular difference between the sines of the two adjacent angles which appear in the tables are not treated as decimals. The product of (A) and (B) is computed without regard to the decimal point of the tabular difference, and the decimal point in the product is then placed by inspection.

sine 30° 0.3′

30°	
′	Sine
0	.50000
1	.50025

Sin 30° 0.3′ =
sin 30° 0′ = .50000
sin 30° 1′ = .50025
Tab Diff. = .00025
(.3) (25) = 7.5

If the given angle were included in the tables, it would occupy a position of .3 of the distance between 30° 0′ and 30° 1′. The value of its sine would, therefore, be .3 of the total change of (25) units, regardless of the decimal points, the sine of the angle will be (.3) (25) or 7.5 units larger than the sine 30° 0′. The proper position to annex the 7.5 is, therefore,

$$\begin{array}{r} .50000 \\ 75 \\ \hline \sin 30° \ 0.3' = .500075 \\ \sin 30° \ 0.3' = .50007 \text{ or } .50008 \end{array}$$

The interpolated value can be no more accurate than the table from which it is derived. The number of digits retained in the interpolated value should not be greater than the number of digits occurring in the table being used. The interpolated value in this case is therefore, assumed to be either .50007 or .50008 which are the closest numerical values of five places representing .500075.

The term tabular difference as used above means the difference in the two values of the sine function corresponding to the two angles in the tables between which the given number lies. Note that when interpolating to find the sine of an angle, the product of (A) and (B) is always added to the value of the sine of the major part of the angle. This is because the value of the sine function increases with increasing magnitude of the angle. This statement has been repeated for emphasis.

The pocess of finding an angle corresponding to a given value of its sine is a reverse process to the procedure just described. However, the solution for such an angle is frequently incorrect, and the following notes and example are included to eliminate any question as to the procedure to be followed.

$\sin\theta = .27011$

15°	
′	Sine
40	.27004
41	.27032

$\sin\theta = .27011$
sin 15° 40′ = .27004
sin 15° 41′ = .27032
Tab. Diff = .00028
sin 15° 40′ = .27004
$\sin\theta$ = .27011
Difference = .00007

It is apparent that the given angle is larger than 15° 40′ by a fraction of a minute which can be expressed as a ratio whose numerator is the amount by which the value of the sine of the given angle exceeds the sine of 15° 40′ and whose denominator is the value by which the value of the sine function increases from 15° 40′ to 15° 41′.

$$\frac{.00007}{.00028} = \frac{7}{28} = \frac{1}{4} = .25$$

Thus the angle whose sine is .27011 is an angle which is .25 of a minute larger than $15° 40'$. Therefore, $\theta = 15° 40.25'$. The answer obtained is stated in degrees and minutes and fractions of a minute as this is the more convenient form in most cases. If, in some instance, it is necessary to state the measure of the angle in degrees, minutes and seconds, the seconds can be readily computed by the use of the interpolation fraction.

For the same example:

$$\frac{.00007}{.00028} \times 60'' = \frac{7}{28} \times 60'' = \frac{1}{4} \times 60'' = 15''$$

$$\theta = 15° 40' 15''$$

The example as given above, can also be written

$$\sin\theta = .27011$$
$$\sin^{-1} = .27011$$

The symbol $\sin^{-1}$ is commonly used to denote "an angle whose sine is"---, whatever value follows. Thus $\sin^{-1} = .5$ or simply $\sin^{-1} .5$ denotes an angle whose sine is .5. A similar notation applies to the use of $\cos^{-1}$, $\tan^{-1}$, etc. It should be understood when using this notation that the $^{-1}$ is not an exponent but simply a part of the new symbol for an angle. The symbol $\sin^{-1}$, $\cos^{-1}$, etc., is also read arcsine, arccosine, etc. The notation as described is satisfactory to use whenever the given angle or angles in question is not greater than 90°. However, for larger angles some ambiguity results as in the example above, $\sin^{-1} .5$ denotes an angle which might be 30°, 150°, etc. In problems of calculus, where radian measure is regularly used for differentiations and integrations, the meaning of the symbol $\sin^{-1}x$ is restricted to the number of radians in the numerically smallest angle whose sine is (x). Somewhat similar agreements are made concerning the use of $\cos^{-1}x$, $\tan^{-1}x$, etc.

An exception to the exponential notation described in the first paragraph occurs in the case of the $^{-1}$ power. It is incorrect, for example, to write $(\sin a)^{-1}$ in the abbreviated form $\sin^{-1}a$ since the latter expression denotes an angle whose sine is (a).

Cosine of an Angle

The cosine of an angle which is given in degrees, minutes, and seconds, is found by first finding the value of the function corresponding to the major part of the angle as defined in terms of degrees and minutes only. To this value of the function is *subtracted* a number obtained by multiplying (A) the tabular difference between the cosines of the two adjacent angles which appear in the tables, one being larger than the given angle, and the other angle being smaller than the given angle by (B) the value of the fractions obtained when the seconds of the given angle are expressed as a fraction of a minute, either as a formal fraction, as number of seconds/60, or as a decimal.

Note that when interpolating to find the cosine of an angle, the product of (A) and (B) is always *subtracted* from the value of the cosine of the major part of the angle. This is because the value of the cosine function decreases with increasing magnitude of the angle.

$$\text{Cosine } 40° 40.2' =$$

40°

	/	Cosine
cos 40° 40.2′	40	.75851
	41	.75832

$\cos 40°\,40' = .75851$
$\cos 40°\,41' = .75832$
Tab Diff. $= .00019$
$(.2)\,(.00019) = .000038 = .00004$
$\cos 40°\,40.2' = .75851 - .00004$
$= .75847$

The process of finding an angle corresponding to a given value of its cosine is a reverse process to the procedure just described.

$$\text{cosine } \theta = 0.86597$$

30°

	/	Cosine
cosine $\theta = 0.86597$	0	.86603
	1	.86588

$\cos 30°\,0' = .86603$
$\cos 30°\,1' = .86588$
Tab Diff. $= .00015$
$\cos 30°\,0' = .86603$
$\cos \theta = .86597$
Diff $= .00006$

It is apparent that the given angle is larger than 30° 0′ by a fraction of a minute which can be expressed as a ratio whose numerator is the amount by which the value of the cosine of the given angle is less than the cosine of 30° 0′, and whose denominator is the value by which the value of the cosine function decreases from 30° 0′ to 30° 1′.

$$\text{or } \frac{.00006}{.00015} = \frac{6}{15} = \frac{2}{5} = .4$$

Thus the angle whose cosine is .86597 is an angle which is .4 of a minute larger than 30° 0′. Therefore $\theta = 30°\,0.4'$.

> Although the process of addition is known to be somewhat simpler than the process of subtraction and, therefore, preferable from a mathematical viewpoint, the above described method of interpolation for the cosine function is always recommended instead of an alternate method which would allow addition processes and involve the use of the next higher function in the table than that corresponding to the major part of the angle as defined in terms of degrees and minutes only.

The additional suggestions, rules, and definitions given for the interpolation of the sine function apply equally to the interpolation of the cosine function. In fact the two procedures are the same except that in the case of the cosine function, the product of (A) and (B) is always subtracted from the value of the cosine of the major part of the angle. This is because the value of the cosine function decreases with increasing magnitude of the angle.

Tangent of an Angle

As already stated, the method in interpolation to find the value of the sine of an angle is similar to the method of interpolating to find the value of the tangent of an angle, since he values of both of these functions increase as the size of the angle increases from 0° to 90°.

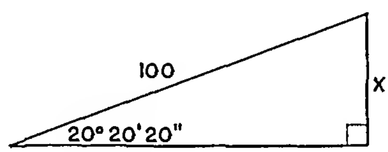
Cotangent, Secant and Cosecant

The trigonometric functions cotangent, scant and cosecant may be called the reciprocal functions since they are the reciprocals of the tangent, cosine and sine respectively. Ordinary tables of the natural trigonometric ratios do not always contain all of the reciprocal functions, but where these are included, interpolation may be performed by the same methods as already described in the preceding paragraphs.

It should be recognized that if a function of an angle increases as the angle increases, then the reciprocal function of the angle will decrease as the angle increases, and vice versa. Consequently, when interpolation is required, observe whether the function being interpolated increases or decreases as the angle increases, and add or subtract as already described for the sine and cosine functions.

SOLUTION OF RIGHT TRIANGLES

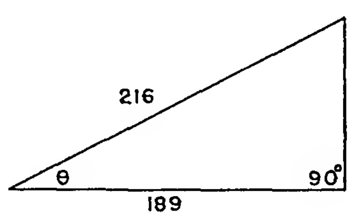
The solution of several right triangles will clearly demonstrate the brevity of the operations as described above.



20°

′	Sine
20	.34748
21	.34775

$$\text{in } 20°\ 20'\ 20'' = \frac{x}{100}$$
$$x = 100\ (\sin 20°\ 20'\ 20'')$$
$$\text{in } 20°\ 20'\ 20'' = \sin 20°\ 20'\ (\text{appr.})$$
$$\text{in } 20°\ 20' = .34748$$
$$x = (100)\ (.34748)$$
$$= 34.748\ (\text{approx.})$$
$$x = 34.75\ (\text{approx.})$$



28°

′	Cosine
57	.87504
58	.87490

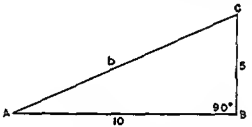
$$\cos\theta = \frac{189}{216} = .87500$$
$$\cos 28°\ 57' = .87504 = .87500 + .00004$$
$$\cos 28°\ 58' = .87490 = .87500 - .00010$$
$$\theta = 28°\ 57'\ (\text{approx.})$$

The true value of the above two answers are, $x = 34.757$ and $\theta = 28°\ 57.28'$ respectively.

It is apparent that the results obtained by using the above approximations may be somewhat in error. The magnitude of this error will vary directly as the amount by

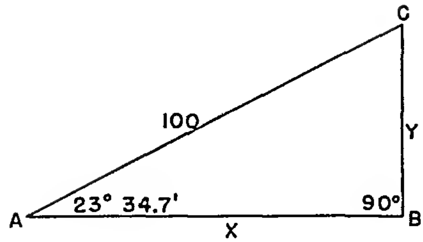
which the given value varies from the tabulated values. But even in the extreme cases, the discrepancies are comparatively small and the results obtained are sufficiently accurate for all ordinary purposes. Where more accurate computations are required, the procedure described under interpolation may be employed.

From the foregoing two examples it is apparent that the solution of right triangles for one or more of the unknown elements may be a simple algebraic process once the relationship among the sides has been determined. In many cases, where only one of a possible two or three elements is to be found the most direct solution is at once observed—the most direct solution being also the most logical solution from a labor and accuracy viewpoint. However, where a triangle is to be completely solved for all sides and all angles it is quite possible that a variety of solutions may be employed in finding one or more of these elements. It is the purpose of the following paragraphs to explain the advantages of employing certain procedures in some instances. These suggestions are not to be interpreted as definite rules, however, as the exact method of solution is left to the user's discretion, which will invariably be tempered by the numerical value of the elements concerned.



$$b^2 = 10^2 + 5^2 = 100 + 25 = 125 \quad \text{(Rule 186)}$$
$$b = \sqrt{125} = 11.18$$
$$\tan A = \frac{5}{10} = .50000$$
$$A = 26° \ 33.92'$$
$$\tan C = \frac{10}{5} = 2.0000$$
$$C = 63° \ 26.13'$$

The value of C could have been found by subtracting the value of A from 90°. This is often done in practical work, and is entirely logical provided that the value of A has been precisely computed. If A is in error the value of C will likewise be incorrect. The above solution guards against this possibility and provides a means by which the two answers may be checked for accuracy. If in solving right triangles the value of the second acute angle is not computed independently as suggested above, but is found by subtracting from 90°, then in solving a problem involving the tangent function, the value of the smaller angle should be the one computed if the results obtained are to be as accurate as possible. This difference in accuracy results from erroneous assumption employed in the interpolation process. The magnitude of this error is a maximum for large angle involving the tangent function. For the sine and cosine function this error is of no consequence.



$$\sin 23°\ 34.7' = \frac{y}{100}$$

$$\begin{aligned} y &= (100)\ (\sin 23°\ 34.7') \\ &= (100)\ (.40000) = 40.00 \end{aligned}$$

$$\cos 23°\ 34.7' = \frac{x}{100}$$

$$\begin{aligned} x &= (100)\ (\cos 23°\ 34.7') \\ &= (100)\ (.91651) = 91.651 \\ C &= 90° - 23°\ 34.7' = 66°\ 25.3' \end{aligned}$$

Check:

$$\begin{aligned} 100^2 &= x^2 + y^2 = 91.65^2 + 40^2 \\ 10000 &= 8400 + 1600 = 10000 \text{ (Slide Rule)} \end{aligned}$$

In this example the value of C is found by subtracting the value of A from 90°. This is the logical procedure inasmuch as the value of A was given and therefore free of any possibility of being in error.

For purposes of demonstrating the value of A was chosen so that its sin would be of such value that when multiplied by the hypotenuse it would produce a rational number which could be conveniently squared. The solution for side (x) is then accomplished by the use of the Pythagorean theorem (Rule 186). This method is sometimes used but is not considered as good practice as the method shown for two reasons, namely:

Only in rare cases will the computed value of (y) be a rational number, or if a rational number, of such magnitude as to be squared by a mental process.

The value of (x) obtained is a computed value and consequently may be in error.

The use of the Pythagorean theorem is, however, recommended as a check on the value of the sides obtained by the method as shown.

In some cases more than one trigonometric function and more than one unknown element is involved in the solution of a given triangle. This requires that more than one equation be written, and that these equations be solved simultaneously.

$$
\begin{array}{l}
(A)\ (\sin 30^\circ) + (B)\ (\sin 60^\circ) = 100 \\
(A)\ (\cos 30^\circ) - (B)\ (\cos 60^\circ) = 0 \\
\hline
.5A + .866B = 100 \\
.866A - .5B = 0 \text{ or } .866A = .5B \\
(.866)\ (.5A) + (.866)\ (.866B) = (.866)\ (100) \\
(.5)\ (.866A) - (.5)\ (.5B) = (.5)\ (0) \\
\hline
.433A + .75B = 86.6 \\
.433A - .25B = 0 \\
\hline
1.00B = 86.6 \\
B = 86.6 \\
.866A = .5B = (.5)\ (86.6) = 43.3 \\
A = 50
\end{array}
$$

$$(L)\ (\sin\theta) = \frac{WV^2}{gR}$$

$$(L)\ (\cos\theta) = W$$

Then

$$\frac{(L)\ (\sin\theta)}{(L)\ (\cos\theta)} = \frac{V^2}{gR}$$

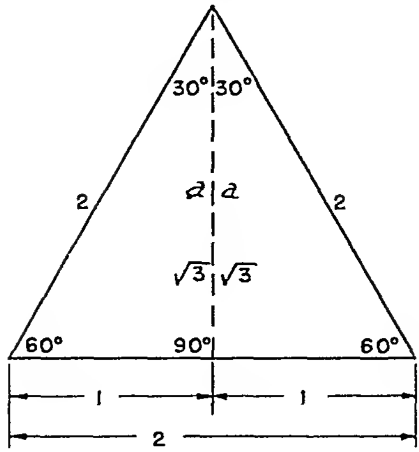
$$\tan\theta = \frac{V^2}{gR}$$

The preceding solutions together with the examples shown on page 137 and 138 are but a few of the many possibilities which are frequently encountered in trigonometric solutions It is recommended that whenever any solution is attempted, the thought is kept in mind to try to obtain the maximum in accuracy with minimum expenditure of labor. After any solution, the results obtained should be checked for accuracy. For this operation, one or more of the geometric relations appearing on page 130 may be employed Also, it is possible to construct a triangle from the given data, using a reasonably large scale for the drawing. The value of the unknown elements may be determined from this figure using a protractor and rule, and the results obtained compared to those obtained by the algebraic computation This graphical check is only approximate, but is very useful in such instances where such a degree of accuracy is permissible

Special Solution of 30°-60° and 45° Right Triangles

Right triangles containing one 30° and one 60° angle (called a 30°-60° right triangle) are so frequently encountered that it is advisable to remember the values of the sin, cos and tan for these two angles Right triangles containing two 45° angles are also of common occurrence for which the common trigonometric functions are often required The various functions for these three angles can easily be memorized by employing a simple fractional notation as now derived.

Ratios of 30° and 60°



The trigonometric functions of 30° and 60° can be found by the use of an equilateral triangle. This may be of any arbitrary size but to simplify the computations which are to follow the dimension of each side is assumed to be *two* units in length in any system of measurement. An altitude drawn to any vertex bisects that angle forming two 30° angles. The side opposite the vertex is also bisected forming two equal segments of unit length. The altitude therefore divides the given equilateral triangle into two identical right triangles which have a hypotenuse two units in length and the shorter side of one unit in length. By the Pythagorean theorem, the length of the remaining side is found to be $\sqrt{3}$ or 1.732 since,

$$2^2 = 1^2 + a^2$$

$$a^2 = 4 - 1 = 3$$
$$a = \sqrt{3} = 1.732\ldots$$

The trigonometric functions of 30° and 60° can now be derived.

$$\text{Sin } 30° = \frac{1}{2} = .500$$

$$\text{Cos } 30° = \frac{\sqrt{3}}{2} = \frac{1.732}{2} = .866\ldots$$

$$\text{Tan } 30° = \frac{1}{\sqrt{3}} = \frac{1}{1.732} = .577\ldots$$

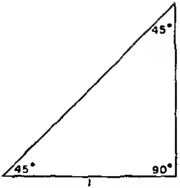
$$\text{Sin } 60° = \frac{\sqrt{3}}{2} = \frac{1.732}{2} = .866\ldots$$

$$\text{Cos } 60° = \frac{1}{2} = .500$$

$$\text{Tan } 60° = \frac{\sqrt{3}}{1} = \frac{1.732}{1} = 1.732\ldots$$

Ratios for 45°

$h =$
$\sqrt{2} = 1\ 414$



The trigonometric functions for 45° can be found by the use of an isosceles triangle. This may be of any arbitrary size, but to simplify the computations which are to follow the dimension of the two equal-length sides is assumed to be *one* unit in length in any system of measurement. By the Pythagorean theorem, the length of the hypotenuse is found to be $\sqrt{2}$ or 1 414 (since $h^2 = 1^2 + 1^2 = 2$ Therefore $h = \sqrt{2}$)

$$\text{Sin } 45° = \frac{1}{\sqrt{2}} = \frac{1}{1.414..} = 707\ldots$$

$$\text{Cos } 45° = \frac{1}{\sqrt{2}} = \frac{1}{1\,414..} = 707..$$

$$\text{Tan } 45° = \frac{1}{1} = 1$$

From the foregoing derivations,

$$\text{Sin } 30° = \frac{1}{2}$$

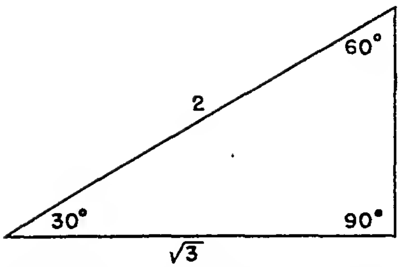
$$\text{Sin } 45° = \frac{1}{\sqrt{2}}$$

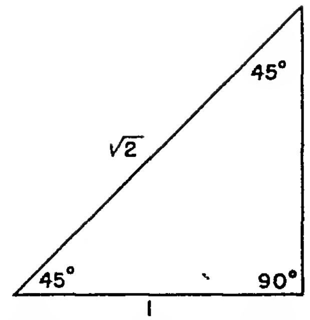
$$\text{Sin } 60° = \frac{\sqrt{3}}{2}$$

The sine of 30° can also be written as $\sqrt{1}/2$, and the sine of 45° can be written as $\sqrt{2}/2$ if both the numerator and the denominator of the original fraction are multiplied order named as $\sqrt{1}/2$, $\sqrt{2}/2$and $\sqrt{3}/2$

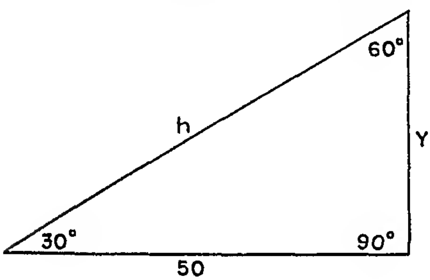
The previously described methods of deriving the sin, cos and tan for 30°, 45° and 60° makes it possible to solve triangles in which these angles occur without the use of a table of the trigonometric functions.

A further simplification in solving 30°-60° and 45° right triangles is possible by employing the theorem of geometry that states that the corresponding sides of similar triangles are proportional. An inspection of the diagrams on page 141 and above shows that the relative lengths of the sides of these triangles can be represented by either small integers or their square roots These values are shown below.





Whenever any given triangle is known to be similar to one of the above types, the length of its sides may be determined, provided that the length of one of the sides is known. If the triangles are similar with only the angles given, there can be an infinite number of triangles of the same shape but of varying areas which will satisfy the stated conditions. Consequently, the values of the sides cannot be found.

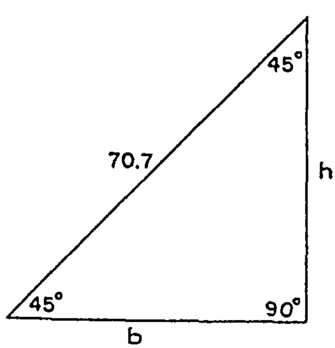


$$\frac{h}{50} = \frac{2}{\sqrt{3}}$$

$$h = \frac{(50)\ (2)}{\sqrt{3}} = \frac{100}{1.732}$$

$$= 57.8 \text{ (approx.)}$$

$$y = \frac{1}{2}(57.8) = 28.9 \text{ (approx.)}$$



$$\frac{70.7}{b} = \frac{\sqrt{2}}{1}$$

$$b = \frac{70.7}{\sqrt{2}} = \frac{70.7}{1.414} = 50$$

$$h = b = 50$$

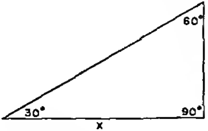
Reciprocal Functions

The trionometric functions cotangent, secant and cosecant may be called the reciprocal functions since they are the reciprocals of the tangent, cosine and sine, respectively. The reciprocal functions are not necessary for the solution of right-triangles, but their use in some cases may simplify the computations involved. Thus, the result obtained by dividing some side by the sine, cosine or tangent of an angle can be more easily found by multiplying that side by the reciprocal of the functions involved. This fact is of importance only when such computations are performed long-hand and is of no consequence when a slide-rule or calculating machine is employed.

Ordinary tables of the natural trigonometric functions do not always include all of the reciprocal functions, but in most cases the cotangent, at least, is represented. This function is employed in the example below in which an operation of division is re-

placed by one of multiplication Similar applications are possible with the other reciprocal functions shown by the solution of the two additional examples on the following page.

Example 1.

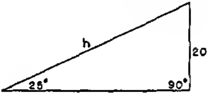


$$\tan 30° = \frac{1000}{x}$$

$$x = \frac{1000}{\tan 30°} = (1000)\left(\frac{1}{\tan 30°}\right) = 1000 \cot 30°$$

$$= (1000)\,(1.732051) = 1732.051$$

Example 2

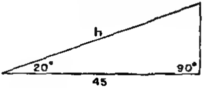


$$\sin 25° = \frac{20}{h}$$

$$h = \frac{20}{\sin 25°} = (20)\left(\frac{1}{\sin 25°}\right) = (20)\ (\text{cosec } 25°)$$

$$= (20\ (2.3662) = 47.324$$

Example 3.



$$\cos 20° = \frac{45}{h}$$

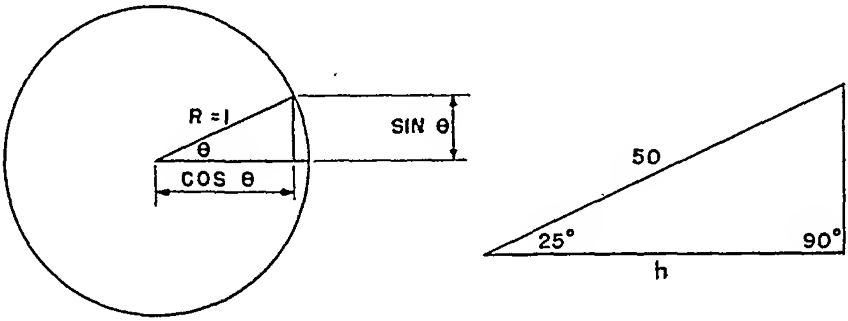
$$h = \frac{45}{\cos 20°} = (45)\left(\frac{1}{\cos 20°}\right) = (45)\ (\sec 20°)$$

$$= (45)\ (1.1034) = 49.653$$

Special Solution of Right Triangles Having the Hypotenuse Given

The frequent occurrence in the study of mechanics of the need for finding the vertical and horizontal components of a given force makes it desirable to know a method of solution with the least amount of effort involved Such a problem is equivalent to finding the other two sides of a right triangle when the hypotenuse and one or both of

the acute angles are given. If one acute angle is given the value of the second acute angle can be obtained by a simple subtraction. Therefore, having one acute angle given is practically the same as though both were given.



An inspection of the diagram on the left indicates that in a circle with a radius of unity the numerical values of the trigonometric ratios of sine and cosine are actually equal to the length of the lines as shown. This fact is established on page 148. The value of the sine and cosine for *any angle whatsoever* is always a decimal fraction as these functions are represented by the sides of a right triangle which has a hypotenuse (radius) equal to unity.

A comparison of the above two figures shows them to be directly comparable, each having the same kind and number of elements given. The hypotenuse of the right triangle in the diagram on the left is 1, while that of the right triangle on the right is equal to 50. If, for an example, θ is arbitrarily assumed to be 25°, then sin θ will have a numerical value corresponding to sin 25°, or .42262. Similarly, cos θ will have numerical value corresponding to cos 25° or .90631. With θ assumed to be 25° the two right-triangles are similar.

Therefore,
$$\frac{50}{1}=\frac{v}{.42262}$$
$$v=(50)\,(.42262)=21.131$$
$$\frac{50}{1}=\frac{h}{.90631}$$
$$h=(50)\,(.906308)=45.3154$$

The equations for finding (v) and (h) can be written directly without first establishing the proportions as shown above if the idea of the unit-radius circle is kept in mind.

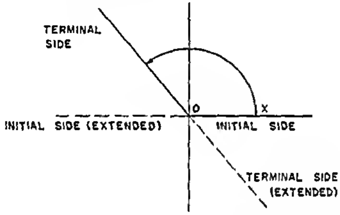
TRIGONOMETRIC FUNCTIONS IN OBLIQUE TRIANGLES

The previous definitions of the trigonometric functions, as the ratios of the lengths of the various sides in a right triangle, becomes confusing when oblique triangles are being solved by the use of special oblique triangle formulas. For this reason an alternate explanation of the functions is presented, in which the definitions are not associated with right triangles, but with any angle. A knowledge of polar coordinates is essential for this explanation.

Polar Coordinates

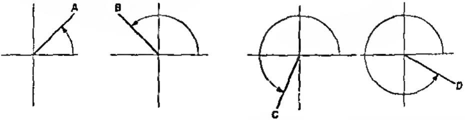
It has already been shown on page 39, that any point in a place can be located by means of Cartesian coordinates.

A second method of locating a point io a plane makes use of angular measurement together with the distance of the point from the *pole*. The pole in the polar coordinate system is the same as the origin in the Cartesian system. In mathematical work, angular measurement commences on the positive side of the X-X axis and increases positively with counter-clockwise rotation about the pole. An angle is therefore bounded by the positive side of the X-X axis, called the *initial side*, and by the *terminal side* which corresponds to a radius which has been rotated about the pole to produce an angle of the desired magnitude



The distance of the point from the pole is determined its *radius vector* or simply *radius*, and is further abbreviated as (r) The radius is always considered a positive quantity when laid off on the terminal side of the angle, and negative if laid off on the terminal side produced (extended) though the pole.

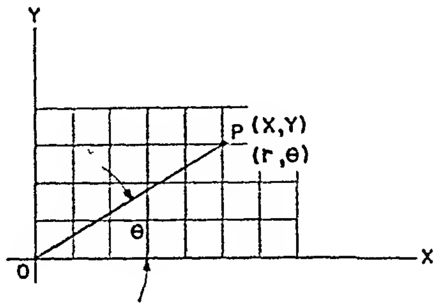
In designating a point, the radius vector is stated first, then the angle. Thus $A(5,45°)$, B (8,135°), C (12,250°), and D (15,330°) represent points in the first, second, third and fourth quadrants, respectively.



To plot a point whose polar coordinates are given, begin by drawing a line on which the radius vector lies—that is a line of indefinite length, but making the specified angle with the initial side. Next, lay off on that line the radius vector using a convenient scale of distances.

Transformation from Polar to Cartesian Coordinates or Vice Versa

If the pole and the initial side of a polar system coincide with the origin and the OX axis of a Cartesian system, the coordinates of any point in the two systems are readily convertible.



θ is the Greek letter Theta

Let the point P have the Cartesian coordinates (x,y) and the polar coordinates (r,θ). Using the trigonometric relations, we have:

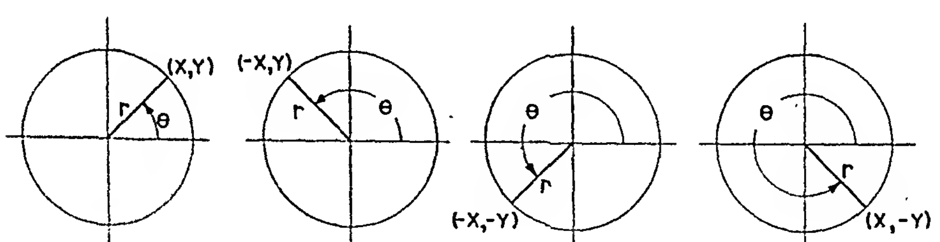
$$x = r\cos\theta$$
$$y = r\sin\theta$$

also,

$$r = \sqrt{x^2 + y^2}$$
$$\text{Tan}\,\theta = y/x$$

These equations enable the transformation of the coordinates of a point from one system to the other whenever such transformation is desired.

By the use of polar coordinates as described above, it is possible to define the trigonometric functions for any angle regardless of its magnitude, that is, whether acute or obtuse.



$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

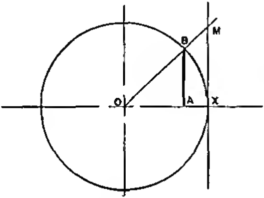
$$\text{cosec}\,\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \text{cotan}\,\theta = \frac{x}{y}$$

The algebraic sign $(+)$ or $(-)$ to be assigned to any particular functions depends upon the quadrant in which the angle lies. For example, the values of both (x) and (y) are positive in the first quadrant, and since the radius vector, (r), is always positive, the values of all six trigonometric functions in the first quadrant are positive.

The value of any function in any other quadrant is determined by the sign of (x) and (y) in that quadrant Thus, the cosine of any angle lying in the second quadrant is negative, since the value of (x) is negative.

GEOMETRIC REPRESENTATION OF THE TRIGONOMETRIC FUNCTIONS

The trigonometric functions can also be represented by the lengths of certain lines in a circle. If the *radius* of the circle is taken as *unity* (one unit in any system) then the trigonometric ratios are actually equal to the length of these lines measured to the same scale as the radius.



$OB =$ Radius $= 1$

Figure 1

With a radius of *unity* and the vertex of the angle at the center, a circle can be described and a tangent drawn at the point (X) where the initial side (OA) cuts the circle The trigonometric functions of sin, cos and tan, are as follows:

$$\sin\theta = AB/OB$$

but, $$OB = 1$$

$$\sin\theta = AB/1 = AB \qquad \text{(Fig. 1)}$$

Definition: The sin of an angle is the perpendicular distance from the point where the terminal side of the angle cuts the circle, to the initial side OA (extended if necessary as for angles in the range 90° to 270°).

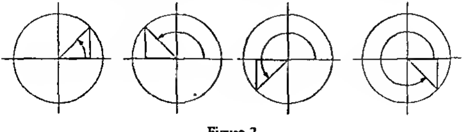


Figure 2

The sin is positive $(+)$ when measured above the X-X axis, and negative $(-)$

when measured below the X-X axis. The sin is therefore always positive when the angle is in the first or second quadrant, and negative in the third or fourth quadrant.

$$\cos\theta = OA/OB = OA/1 = OA \qquad \text{(Fig. 1)}$$

> *Definition:* The cosine of an angle is the projection of the terminal side on the X-X axis, or as might be stated, the distance from the center, (origin or pole), to the point where the perpendicular line representing the sin cuts the initial side (extended if necessary, as for angles in the range 90° to 270°).

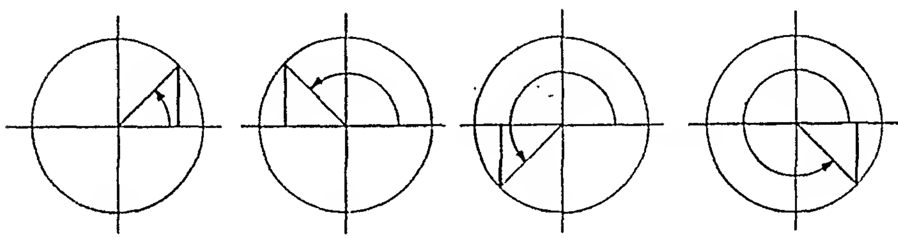


Figure 3

The cosine is positive (+) when measured to the right of the Y-Y axis, and negative (—) when measured to the left of the Y-Y axis. The cosine is therefore always positive when the angle is in the first or fourth quadrants, and negative when in the second or third quadrants.

> *Definition:* The tangent of an angle is the distance along the line tangent to the circle, from the point where the initial side cuts the circle to the point where the terminal side of the angle cuts the tangent line. (Extend the terminal side of the angle if necessary, as for angles in the range 90° to 270°).

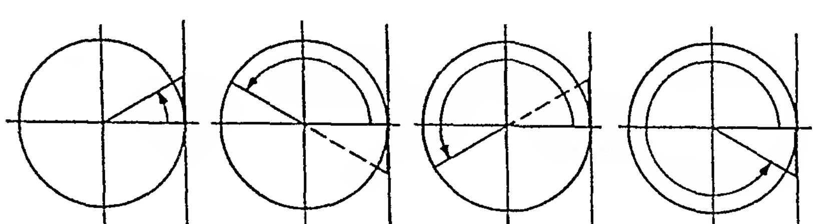


Figure 4

The tangent of an angle is positive (+) when measured above the X-X axis, and negative (—) when measured below the X-X axis. The tangent of an angle is therefore always positive when the angle is in the first or third quadrant, and negative in the second or fourth quadrant. It may be easier for some to remember that the tan $\theta =$ sin

θ/cos θ, (Rule 224) and from this relationship, the prefix sign for the tan of any angle can be readily determined by substituting the signs for the sin and the cos for this same angle.

VALUE OF THE FUNCTIONS OF OBTUSE ANGLES

It is sometimes necessary to find the value of one or more of the trigonometric functions for angles greater than 90°. The ordinary tables of the natural trigonometric functions include only those angles from 0° to 90°. However, it can be shown that the sin, cos, tan, etc., of any angle over 90° is the same in magnitude as some acute angle which will be included in the tables The rule for finding this equivalent acute angle is as follows:

> Take the difference between the given obtuse angle and 180° or 350°, whichever gives an acute angle, and prefix the proper sign ($\pm$) to the value of the function according to the quadrant in which the original (obtuse) angle lies. (Rule 214)

If an angle (such as 225°) after being subtracted from 360° does not yield an acute angle, then a second subtraction is performed to find the difference between 180° and the difference as already found between 360° and the given angle.

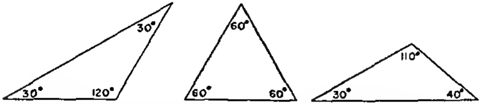
$$360° - 225° = 135°$$
$$180° - 135° = 45°$$

The validity of the rule as stated above should be apparent from an examination of figures 2, 3 and 4. It should also be evident that any particular trigonometirc function of two angles will be equal in magnitude if one or more of the following conditions are fulfilled:

1. That the angles differ by 180°.
2. That the angles differ by equal amounts from the X-X axis (from 0° or 180°).
3. That the angles differ by equal amounts from the Y-Y axis (from 90° to 270°).

OBLIQUE TRIANGLES SOLVED AS RIGHT TRIANGLES

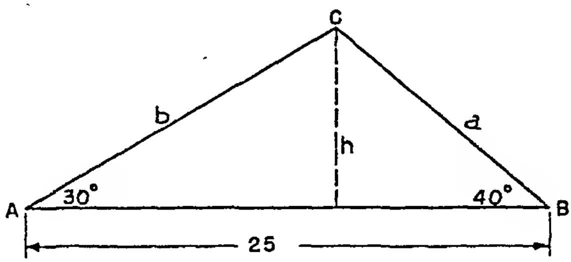
Oblique triangles are triangles which do not contain a right triangle (90°). Such triangles may or may not contain one angle greater than a right angle as shown below.



The solution of oblique triangles, in general, is most easily accomplished by the use of special formulas of which the cosine law and the sine proportion are the most commonly used. These are demonstrated by examples in later paragraphs. However, there

are many instances in which oblique triangles can be solved by the methods previously described for the solution of right triangles. A few of these are included for the purpose of demonstration. When solving oblique triangles as right triangles it is usually possible to employ any one of several different procedures.

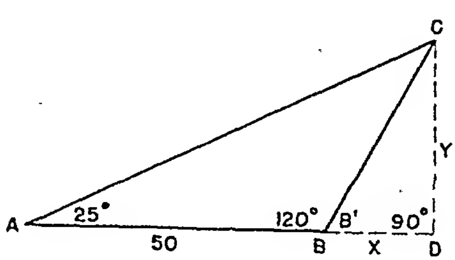
Two Angles and Any Side. (This is equivalent to having all three angles and any side given.)



Draw CD or (h) perpendicular to AB forming right-triangle ACD and CDB.

$$
\begin{aligned}
h &= b \sin 30^\circ \\
h &= a \sin 40^\circ \\
b \sin 30^\circ &= a \sin 40^\circ \\
b \sin 30^\circ - a \sin 40^\circ &= 0 \\
\underline{b \cos 30^\circ + a \cos 40^\circ} &\underline{= 25} \\
.5\,b \quad - .64279\,a &= 0 \qquad \text{or } b = 1.28558\,a \\
\underline{.86603\,b + .76604\,a} &\underline{= 25} \\
(.86603)\,(1.28558\,a) + .76604\,a &= 25 \\
1.11335\,a + .76604\,a &= 25 \\
1.87939\,a &= 25 \\
1.88\,a &= 25 \\
a &= 13.297 = 13.30 \text{ (approx.)} \\
b &= (1.28558)\,(13.30) \\
&= 17.10 \text{ (approx.)} \\
C &= 180^\circ - (30^\circ + 40^\circ) = 110^\circ
\end{aligned}
$$

Two Angles and Any Side. (This is equalent to having all three angles and any side given.)



Extend AB to D forming right-triangle ACD. Denote BD by (x) and CD by (y).

$$B' = 180° - 120° = 60°$$

$$\tan 25° = \frac{y}{50+x} \quad \text{or} \quad y = .46631\ (50+x)$$

$$\tan 60° = \frac{y}{x} \quad \text{or} \quad y = 1.7320x$$

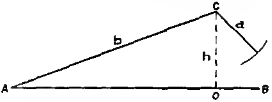
$$\begin{aligned}
1.7320x &= .46631\ (50+x) = 23.3155 + .46631x \\
1.7320x - .46631x &= 23.3155 \\
1.26569x &= 23.3155 \\
x &= 18.42 \text{ (approx.)} \\
y &= (1.7320)\ (18.42) \\
&= 31.90 \text{ (approx.)}
\end{aligned}$$

$$BC = \frac{31.90}{\sin 60°} = \frac{31.90}{.86603} = 36.836 = 36.84\ldots$$

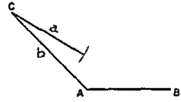
$$AC = \frac{31.90}{\sin 25°} = \frac{31.90}{.42262} = 75.48\ldots$$

$$\begin{aligned}
C &= 180° - (25° + 120°) \\
&= 180° - 145° \\
&= 35°
\end{aligned}$$

Two sides and an angle opposite one of them—Ambiguous Case. An oblique triangle which has only two sides and an angle opposite one of them given is known as the ambiguous case, inasmuch as there may be no solution, one solution, or two solutions



ACUTE ANGLE



OBTUSE ANGLE

In the oblique triangle shown above, assume that the two sides and the angle which are given are the sides (a) and (b), and the angle A which is opposite side (a). Angle A may be either an acute or an obtuse angle

To get any idea of the different types of triangles which may be constructed from the given elements, it is convenient to think of any two of the given elements as fixed, and vary only the magnitude of the third

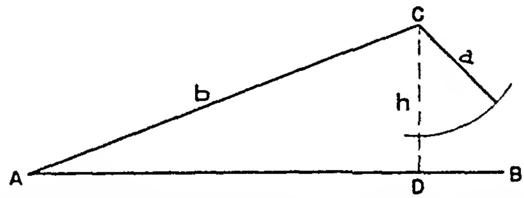
Consequently, it is assumed that angle A and side (b) remain unchanged. For the present discussion, angle A is considered ao acute angle.

A triangle is constructed from the given data by first drawing side (b) and angle A Side (b) is also lettered AC. At end C an arc is swung using the length of side (a) as a radius. In order for side (a) to close the triangle, this side should be long enough to reach the opposite side at some point, B. By drawing CD perpendicular to side AB, a right triangle, ADC is formed. The length of CD, or (b), is:

$$CD = b = b \sin A$$

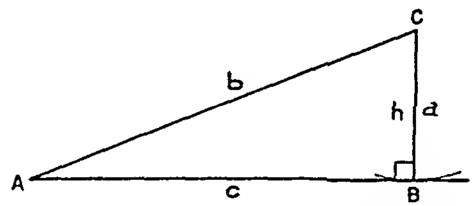
An examination of the figure above shows that the following possibilities may occur:

1.



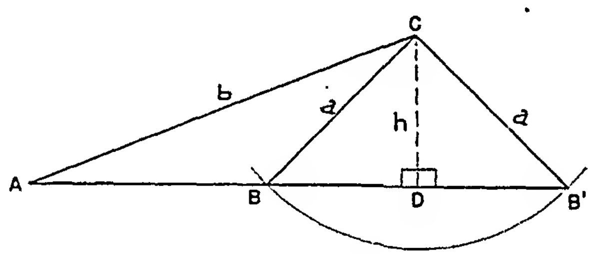
If side (a) is less than (b), it will not be long enough to reach the line ADB, in which case there can be no triangle.

2.



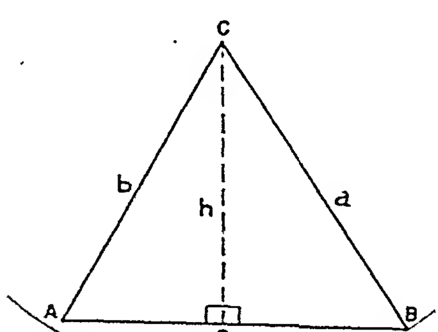
If side (a) equals side (b), then (a) coincides with (b), and the triangle is a right triangle with the right angle at B.

3.



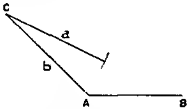
If side (a) is greater than (b), but less than (b), there will be two oblique triangles, ABC and $AB'C$.

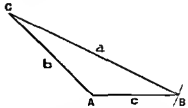
4.



If side (a) is either greater than or equal to (b), it will be greater than (b), and there is only one triangle, ABC.

Now if angle A is obtuse, and side (b) remains unchanged as before, the following additional possibilities may occur:

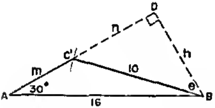


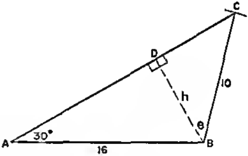


5. If side (a) is either less than or equal to (b), there will be no triangle.
6. If side (a) is greater than (b), there will be only one triangle.

It is apparent that if two sides and an angle opposite one of them is given, there may be no solution, one solution, or two solutions, all depending on the relative magnitudes of the given elements In order that there may be two solutions, the given angle must be acute, and the side opposite it must be less than the side adjacent.

Two Sides and an Angle Opposite One of Them—Ambiguous Case.





Draw BD or (h) perpendicular to AC' and AC forming right triangles ADB.

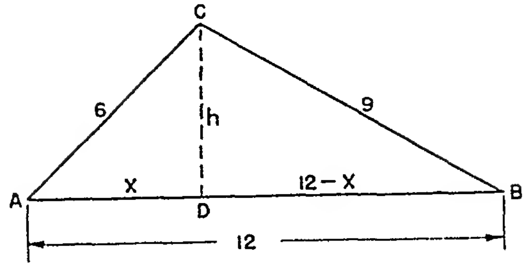
In triangle ABC'

$$h = 16 \sin 30° = 8$$
$$(m+n)^2 = 16^2 - 8^2 = 256 - 64 = 192$$
$$AD = (m+n) = \sqrt{192} = \pm 13.86$$
$$C'D = n = \sqrt{10^2 - 8^2} = \sqrt{36} = 6$$
$$AC' = m = 13.86 - 6 = 7.86$$
$$\cos\theta = \frac{8}{10} = .80000$$
$$\theta = 36° \, 52.18'$$
$$B = 90° - 30° - 36° \, 52.18'$$
$$B = 23° \, 7.82'$$
$$C' = 180° - 30° - 23° \, 7.82'$$
$$C = 126° \, 52.18'$$

In triangle ABC

$$AC = AD + 6 = 13.86 + 6 = 19.86$$
$$B = 90° - 30° + 36° \, 52.18'$$
$$B = 96° \, 52.18'$$
$$C = 180° - 30° - 96° \, 52.18'$$
$$C = 53° \, 7.82'$$

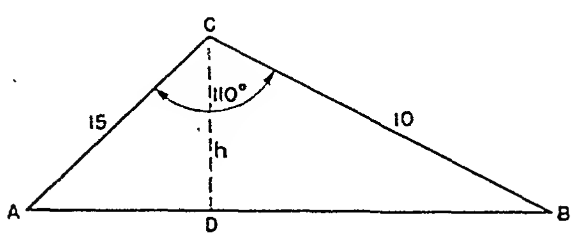
Three sides given:



Draw CD or (h) perpendicular to AB forming right-triangles ACD and CDB.

$$6^2 = h^2 + x^2 \qquad 9^2 = h^2 + (12 - x)^2$$
$$h^2 = 36 - x^2 \qquad h^2 = 81 - (12 - x)^2$$
$$36 - x^2 = 81 - 144 + 24x - x^2$$
$$99 = 24x$$
$$x = 4.125$$
$$12 - x = 12 - 4.125 = 7.875$$
$$\cos A = \frac{4.125}{6} = .68750$$
$$A = 46° \ 34.05'$$
$$\cos B = \frac{7.875}{9} = .87500$$
$$B = 28° \ 57.29'$$
$$= 180° - (A + B)$$
$$= 180° - (46° \ 34.05' + 28° \ 57.29')$$
$$= 180° - 75° \ 31.34' = 104° \ 28.66'$$

Two Sides and the Included Angle

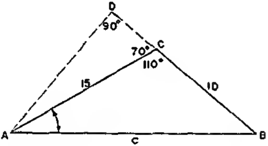


Draw CD or (h) perpendicular to AB forming right-triangles ACD and CDB.

$$A + B = 180° - 110° = 70°$$
$$A = 70° - B$$
$$\therefore (15)\ (\sin A) = h \qquad (10)\ (\sin B) = h$$
$$(15)\ (\sin A) = (10)\ (\sin B)$$
$$15 \sin\ (70° - B) = 10 \sin B$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y \qquad \text{(Rule 231)}$$
$$15(\sin 70^\circ \cos B - \cos 70^\circ \sin B) = 10 \sin B$$
$$15(.93969 \cos B - .34202 \sin B) = 10 \sin B$$
$$14.09535 \cos B - 5.13030 \sin B = 10 \sin B$$
$$14.09535 \cos B = 10 \sin B + 5.13030 \sin B$$
$$14.09535 \cos B = 15.13030 \sin B$$
$$\frac{14.09535 \cos B}{15.13030 \cos B} = \frac{15.13030 \sin B}{15.13030 \cos B}$$
$$0.93160 = \tan B$$
$$B = 42^\circ\, 58.31'$$
$$A = 180^\circ - (110^\circ + 42^\circ\, 58.31')$$
$$= 180^\circ - 152^\circ\, 58.31'$$
$$= 27^\circ\, 1.69'$$
$$AB = AD + DB$$
$$= (15)(\cos 27^\circ\, 1.69') + (10)(\cos 42^\circ\, 58.31')$$
$$= (15)(.89078) + (10)(.73169)$$
$$= 13.3619 + 7.3169 = 20.6788$$
$$= 20.68 \text{ (approx.)}$$

Two Sides and the Included Angle
Alternate Solution



Extend CB to D forming right-triangle ABD

$$BD = (15)(\cos 70^\circ) + 10 = (15)(.34202) + 10$$
$$= 5.13030 + 10 = 15.13030$$
$$= 15.13 \text{ (approx.)}$$

$$AD = (15)(\sin 70^\circ) = (15)(.93969) = 14.09535$$
$$= 14.10 \text{ (approx.)}$$

$$\tan B = \frac{14.09535}{15.13030} = .93160$$
$$B = 42^\circ\, 58.31'$$
$$A = 180^\circ - (42^\circ\, 58.31') + 110^\circ) = 180^\circ - 152^\circ\, 58.31'$$
$$= 27^\circ\, 1.69'$$
$$c^2 = (BD)^2 + (AD)^2 = (15.13)^2 + (14.10)^2 = 228.92 + 198.81 = 427.73$$
$$c = \sqrt{427.73} = 20.68 \text{ (approx.)}$$

OBLIQUE TRIANGLES SOLVED BY SPECIAL FORMULAS

In many cases oblique triangles can be solved by the same methods used in the solution of right triangles. Such a procedure has been described and a variety of examples have been solved to show the simplicity of the method in some cases and the complexity of the solution in other instances. Another conclusion which should be drawn from these examples is that it is often necessary to use computed values in succeeding computations, a policy to be avoided whenever possible.

The solutions of oblique triangles by the use of special formulas seems to be more logical than the searching for relationships which will permit the solution of the problem as a series of right triangles. Furthermore, the use of these formulas is a more direct solution in most cases, and the necessity of using computed values is less frequently required, if at all.

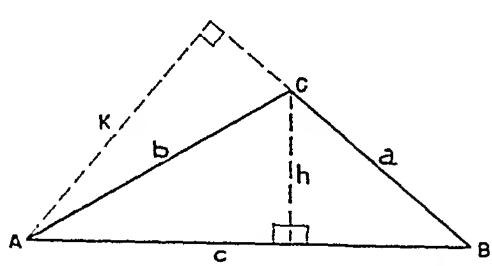
Sine Proportion

Of several formulas applying to oblique triangles, only the sine portion and the cosine law will be described for solutions in which computations are to be made using the slide rule, long-hand, or calculating machine. For the solution of oblique triangles using logarithms, the cosine law is not a practical solution. For this reason, an additional formula adapted to logarithmic computations is stated and proved. This additional method is known as the tangent law.

The sine proportion is used to solve oblique triangles when only the following combinations of angles and sides are known:

1. Two angles and any side. (This is the equivalent of having all the angles and any side given).
2. Two sides and an angle opposite one of them.

The derivation of the sine proportion is as follows:



$$h = b \sin A$$
$$h = a \sin B$$
$$b \sin A = a \sin B$$

or, $$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$k = c \sin B$$
$$k = b \sin C$$
$$c \sin B = b \sin C$$

or, $$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Where (a) is the side opposite angle A, (b) is the side opposite angle B, and (c) is the side opposite angle C.

The sine proportion may be stated.

The ratio of the length of any side to the sine of the opposite angle is a constant. (Rule 215)

also,

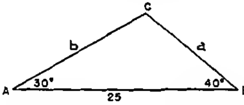
$$\frac{a}{\sin A}=\frac{b}{\sin B}$$

$$\frac{a}{\sin A}=\frac{c}{\sin C}$$

$$\frac{b}{\sin B}=\frac{c}{\sin C}$$

It is evident that if any three of the elements in one of these proportions are given, the fourth element can be found Then, using the value of the element thus found in another proportion, the remaining element can be found, completing the solution.

Two angles and any side. (This is equivalent to three angles and any side).



$$C=180°-(30°+40°)=180°-70°=110°$$
$$\sin 110°=\sin(180°-110°)=\sin 70°=.939693$$

$$\frac{a}{\sin 30°}=\frac{25}{\sin 110°}$$

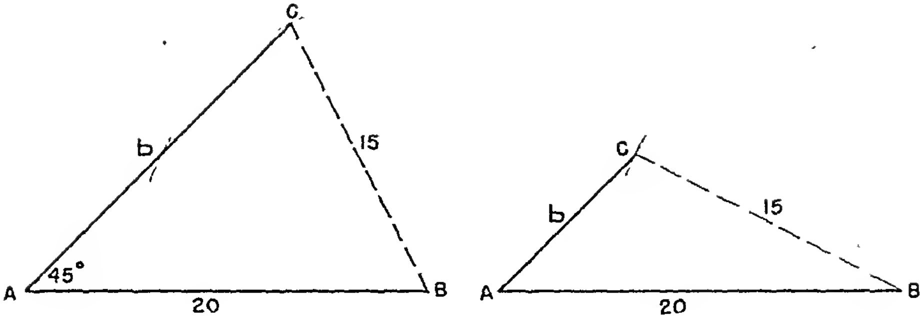
$$a=\frac{(.5)(25)}{.93969}=\frac{12.5}{.93969}=13.30 \text{ (approx.)}$$

$$\frac{b}{\sin 40°}=\frac{25}{\sin 110°}$$

$$b=\frac{(.64279)(25)}{.93969}=17.10 \text{ (approx.)}$$

Two sides and an angle opposite one of them.—Ambiguous Case

An oblique triangle which has only two sides and an angle opposite one of them given is known as the ambiguous case inasmuch as there may be either no solution, one solution or two solutions. These possibilities are explained on page 152. Whether there is no solution, one solution, or two solutions can always be determined by making a careful construction of the triangle from the given elements.



$$\frac{15}{\sin 45^\circ} = \frac{20}{\sin C}$$

$$\frac{15}{.70711} = \frac{20}{\sin C}$$

$$15 \sin C = 20 \times .70711 = 14.1422$$

$$\sin C = \frac{14.1422}{15} = .94281$$

$$C = 70^\circ\ 31.78'$$

Since the value of angle C is determined by means of its sine, it may have two values, one angle being less than 90° and the other angle greater than 90°, and their sum being 180°. Therefore, there *may* be two triangles drawn from the given elements. In one triangle, angle C is $70^\circ 31.78'$ as determined above, and in the other triangle, angle C is $180^\circ - 70^\circ 31.78'$ or $109^\circ 28.22'$.

The value of angle B will likewise vary depending on whether angle C is $70^\circ 31.78'$ or $109^\circ 28.22'$. Consequently, the value of side (b) will also have two values, since side (b) is determined by the sin proportion in which the value of angle B is used.

$$B = 180^\circ - 45^\circ - 70^\circ\ 31.78' = 64^\circ\ 28.22'$$

$$\frac{b}{\sin 64^\circ\ 28.22'} = \frac{15}{\sin 45}$$

$$\frac{b}{.90236} = \frac{15}{.70711}$$

$$.70711b = 13.5354$$

$$b = 19.14 \text{ (approx.)}$$

$$B = 180^\circ - 45^\circ - 109^\circ\ 28.22' = 25^\circ\ 31.78'$$

$$\frac{b}{\sin 25^\circ\ 31.78'} = \frac{15}{\sin 45^\circ}$$

$$\frac{b}{.43098} = \frac{15}{.70711}$$

$$.70711b = 6.4647$$

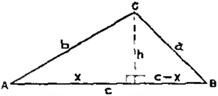
$$b = 9.14 \text{ (approx.)}$$

Cosine Law

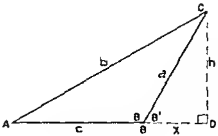
By referring to the sine proportion, it is evident that the sine proportion may be employed whenever a sufficient number and kind of elements are given so that a proportion involving only one unknown can be established between any two of the ratios. This requirement precludes the use of the sine proportion when either of the following combination of elements are all that are given.

1. Three sides
2. Two sides and the included angle.

The derivation of the cosine law is as follows.



ACUTE ANGLE



OBTUSE ANGLE

$$h^2 = b^2 - x^2, \text{ and } b^2 = a^2 - (c - x)^2$$
$$= a^2 - c^2 + 2cx - x^2$$
$$b^2 - x^2 = a^2 - c^2 + 2cx - x^2$$
$$a^2 = b^2 + c^2 - 2cx$$

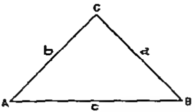
$$\cos A = \frac{x}{b}, \text{ or } x = b \cos A$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$h = b \sin A$$
$$x = b \cos A - c$$
$$a^2 = b^2 + x^2 = (b \sin A)^2 + (b \cos A - c)$$
$$= b^2 \sin^2 A + b^2 \cos^2 A - 2bc \cos A + c^2$$
$$= b^2 (\sin^2 A + \cos^2 A) + c^2 - 2bc \cos A$$
$$\text{but, } \sin^2 A + \cos^2 A = 1$$
$$a^2 = b^2 + c^2 - 2bc \cos A$$

The cosine law may be stated

The square of any side of a triangle is equal to the sum of the squares of the other two sides diminished by twice the product of the two sides multiplied by the cosine of their included angle. (Rule 216)

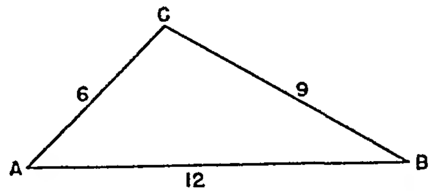


$$a^2 = b^2 + c^2 - 2bc \operatorname{Cos} A$$
Similarly, $b^2 = a^2 + c^2 - 2ac \operatorname{Cos} B$
Similarly, $c^2 = a^2 + b^2 - 2ab \operatorname{Cos} C$

It is not recommended that the equations as written be memorized because the letters assigned to the sides and angles may be entirely different in an actual problem. Several applications of the law are sufficient to fix the law in mind, especially if the similarity of the law with the Pythagorean theorem is recognized. Bearing in mind that within a triangle the cosine of any angle greater than 90° is negative, the term $-2bc \cos A$, and so on, becomes positive if the included angle is over 90°. This makes the corre-

sponding value of a^2 larger than it would be if $A = 90°$. Similarly, if the included angle is less than 90°, the term $-2bc \cos A$ remains negative, making the corresponding value of a^2 less than it would be if $A = 90°$. For an angle of exactly 90° the term $-2bc \cos A$ becomes zero inasmuch as $\cos 90° = 0$, and the cosine law for the particular angle is the same as the Pythagorean theorem. (Rule 186)

Three sides given:



$$81 = 36 + 144 - (2)\ (6)\ (12)\ (\text{Cos}\, A)$$

$$\text{Cos}\, A = \frac{99}{144} = \frac{11}{16} = .68750$$

$$A = 46°\ 34.05'$$
$$36 = 144 + 81 - (2)\ (12)\ (9)\ (\text{Cos}\, B)$$

$$\text{Cos}\, B = \frac{189}{216} = \frac{7}{8} = .87500$$

$$B = 28°\ 57.29'$$
$$144 = 36 + 81 - (2)\ (6)\ (9)\ (-\text{Cos}\, C)$$

$$\text{Cos}\, C = \frac{27}{108} = \frac{1}{4} = .25000$$

$$C = 104°\ 28.64' \qquad \text{(See explanation below)}$$

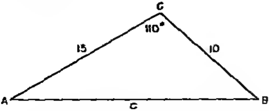
It should be observed that, by using the cosine law throughout the solution, the need for using computed values at any time is eliminated.

However one feature is involved which is troublesome if not understood. An inspection of the trigonometric tables shows that the angle which has a cosine of .25000 is not 104° 28.64′, but 75° 31.36′. Furthermore, the cosine of any angle in a triangle greater than 90° is always negative. Since the value obtained for Cos C was positive, it should be an acute angle. An inspection of the given problem shows angle C to be greater than 90°, a fact easily proved by the Pythagorean theorem, or by subtracting the sum of A and B from 180°. If the unknown value of Cos C had not been entered into the equation as a negative value, the result of the solution would have been $-.25000$ and have indicated that the angle was obtuse. This may be the better procedure, but since the value of the cosine of C was known to be negative, it was entered accordingly. The point of importance is not whether to enter cos C as + or −, but to know whether the angle involved is less or greater than 90°. The angle corresponding to the computed function will be read from the tables as an acute angle. Finding the supplementary angle, if required, is then obtained by a simple subtraction of the acute angle from 180°.

Several variations of the above solution may be employed as shown in the following example. In this instance it is apparent that the sine proportion will not solve the

triangle as given, but is helpful after one of the unknown elements has been found by the cosine law.

Two sides and the included angle:



$$\cos 110^\circ = -\cos(180^\circ - 110^\circ) = -\cos 70^\circ = -.34202$$
$$c^2 = 225 + 100 - (2)(15)(10)(-.34202)$$
$$c^2 = 225 + 100 + 102.606 = 427.606$$
$$c = 20.678\ldots = 20.68 \text{ (approx.)}$$
$$\sin 110^\circ = \sin(180^\circ - 110^\circ) = \sin 70^\circ = .93969$$

$$\frac{10}{\sin A} = \frac{20.68}{.93969}$$

$$\sin A = \frac{(10)(.93969)}{20.68} = .45440$$

$$A = 27^\circ\, 1.58'$$

$$\frac{15}{\sin B} = \frac{20.68}{.93969}$$

$$\sin B = \frac{(15)(.93969)}{20.68} = .68159$$

$$B = 42^\circ\, 58.10'$$

The above method of solution is more direct than if the cosine law had been employed to find the remaining elements The use of the computed value for side (b), or 20.68 is unavoidable, regardless of the method employed.

The most common error in applying the cosine law is the failure to consider that the cosine of any angle in a triangle greater than 90° is negative. This causes the term $(-4xy)(\cos Z)$ to become $(-4xy)(-\cos Z)$ or $+4xy \cos Z$. After the solution is believed to be complete, an inspection to see that the greatest sides are opposite the largest angles often shows an error to exist

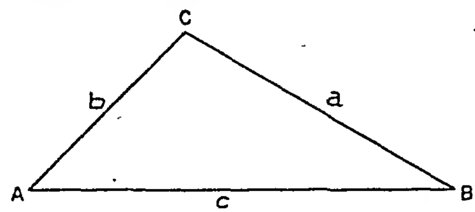
Law of Tangents

The sine proportions and the cosine law will solve any oblique triangle provided that a minimum of three elements are given, one of which must be a side. For slide rule, long hand, or calculating machine computations, these formulas are entirely satisfactory and should be used. However, if the triangles are to be solved by the use of logarithms, the cosine law is not a practical solution, since it is not expressed in terms of either ratios or products, and is therefore not adapted to use with logarithms For this reason, the tangent law is stated and proved

This formula is adapted to logarithmic computation and may be used to solve one of the types of oblique triangles solved by the cosine law, that is when two sides and the

included angle are given. The remaining case covered by the cosine law, three sides only, may be solved by logarithms by using a set of formulas, known as the half-angle formulas, which are tabulated on page 169.

The derivation of the tangent law is as follows:



$$\frac{a}{\sin A} = \frac{b}{\sin B} \text{ or } \frac{a}{b} = \frac{b}{\sin B} \qquad \text{(Rule 215)}$$

$$\frac{a}{b} - 1 = \frac{\sin A}{\sin B} - 1$$

$$\frac{a-b}{b} = \frac{\sin A - \sin B}{\sin B}$$

Similarly,
$$\frac{a+b}{b} = \frac{\sin A + \sin B}{\sin B}$$

Dividing,
$$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}$$

But,
$$\sin A - \sin B = 2 \cos \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B) \qquad \text{(Rule 249)}$$

$$\sin A + \sin B = 2 \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B) \qquad \text{(Rule 248)}$$

Dividing,
$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)}{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}$$

$$\frac{a-b}{a+b} = \cot \tfrac{1}{2}(A+B) \tan \tfrac{1}{2}(A-B)$$

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}$$

Again,
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Therefore,
$$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}$$

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}$$

Similarly,
$$\frac{a-c}{a+c} = \frac{\tan \frac{1}{2}(A-C)}{\tan \frac{1}{2}(A+C)}$$

$$\frac{b-c}{b+c} = \frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)}$$

When the first angle in the equation is greater than the second angle, the above formulas are used. To avoid negative quantities when the second angle is greater than the first, the formulas may be rewritten.

$$\frac{b-a}{b+a}=\frac{\tan \frac{1}{2}(B-A)}{\tan \frac{1}{2}(B+A)}$$

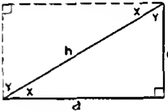
$$\frac{c-a}{c+a}=\frac{\tan \frac{1}{2}(C-A)}{\tan \frac{1}{2}(C+A)}$$

$$\frac{c-b}{c+b}=\frac{\tan \frac{1}{2}(C-B)}{\tan \frac{1}{2}(C+B)}$$

The law of tangents may be stated:
The difference between two sides of a triangle is to their sum as the tangent of half the difference between the opposite angles is to the tangent of half the sum of these angles. (Rule 217)

TRIGONOMETRIC FORMULAS

There are a great number of formulas which are used at different times in solving trigonometric problems A few of the most useful of these are derived in the following pages. Additional formulas are tabulated to provide a convenient reference.



$$x+y=90^\circ$$
$$y=90^\circ-x$$

$$\sin x=\frac{o}{h}$$

$$\cos y=\frac{h}{o}$$

therefore, $\sin x=\cos y=\cos(90^\circ-x)$

(218). This relationship may be stated. The sine of any acute angle is equal to the cosine of its complementary angle
Also, $\cos x=\sin y=\sin(90^\circ-x)$

(219). This relationship may be stated· The cosine of any acute angle is equal to the sine of its complementary angle.
Similarly,

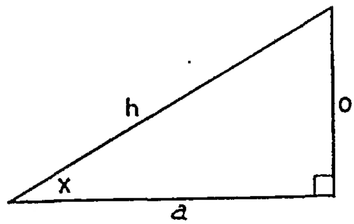
(220). $\tan x=\cot y=\cot(90^\circ-x)$

(221). $\operatorname{cosec} x=\sec y=\sec(90^\circ-x)$

(222). $\sec x=\operatorname{cosec} y=\operatorname{cosec}(90^\circ=x)$

(223). $\cot x=\tan y=\cot(90^\circ-x)$

Relations Between the Functions of One Angle



$$\sin x = \frac{o}{h}$$

$$\cos x = \frac{a}{h}$$

$$\frac{\sin x}{\cos x} = \frac{o/h}{a/h} = \frac{o}{a}$$

but, $$\tan x = \frac{o}{a}$$

(224). Therefore, $$\frac{\sin x}{\cos x} = \tan x$$

(225). $$\frac{\cos x}{\sin x} = \cot x$$

$$o^2 + a^2 = h^2$$

$$\frac{o^2}{h^2} + \frac{a^2}{h^2} = \frac{h^2}{h^2} = 1$$

but, $$\frac{o}{h} = \sin x \qquad \frac{o^2}{h^2} = \sin^2 x$$

$$\frac{a}{h} = \cos x \qquad \frac{a^2}{h^2} = \cos^2 x$$

(226). Therefore, $$\sin^2 x + \cos^2 x = 1$$

To indicate a power of a trigonometric function, it is customary to apply the exponent directly to the function, rather than to write the exponent after the angle. Thus,

$$\sin^2 x \text{ means } (\sin x)^2.$$

The expression $\sin x^2$ would mean the sine of an angle whose number of units is the square of the number in angle (x).

Relations Between the Sum and Difference of Two Angles

The sum of any wo angles such as (x) and (y) can be denoted as (s), and the difference between the same two angles can be denoted as (d). Assume that (x) is the larger angle.

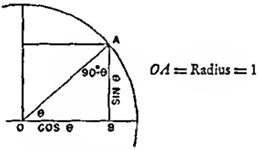
$$\begin{aligned} x+y&=s\\ x-y&=d\\ \hline 2x\quad &=s+d\\ x&=\tfrac{1}{2}(s+d)\end{aligned}$$

(227). or, $$x=\tfrac{1}{2}(x+y)+\tfrac{1}{2}(x-y)$$

$$\begin{aligned} x+y&=s\\ x-y&=d\\ \hline 2y&=s-d\\ y&=\tfrac{1}{2}(s-d)\end{aligned}$$

(228). or, $$y=\tfrac{1}{2}(x+y)-\tfrac{1}{2}(x-y)$$

By constructing a circle with a radius (OA) of unity, and employing the definitions of sin and cos as the lengths of certain lines within that circle, the three preceding relationships may be verified.



(224). $\tan\theta = AB/OB = \sin\theta/\cos\theta$

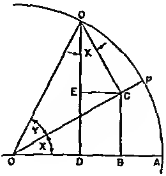
(218). $\sin\theta = \cos(90° - \theta)$

(219). $\cos\theta = \sin(90° - \theta)$

$\sin^2\theta + \cos^2\theta = (OB)^2 = 1^2 = 1$

(226). $\sin^2\theta + \cos^2\theta = 1$

In the figure below, let (x) and (y) be any two acute angles each of which, as well as their sum, is less than 90°. QC is perpendicular to OP. BC and DQ are perpendicular to OA, and EC is parallel to OA. The radius of the circle (OA, OP, or OQ) is equal to unity.



$$x + y = AOQ$$

$$\text{angle } EQC = x;\ OC = \cos y;\ CQ = \sin y$$

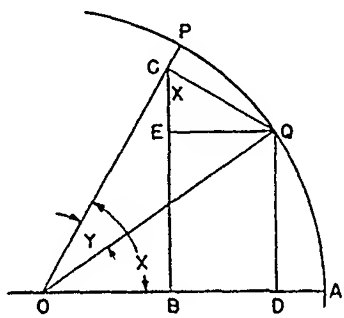
$$\sin(x+y) = DQ = BC + EQ$$
$$= OC \sin BOC + CQ \cos EQC$$
$$= \cos y \sin x + \sin y \cos x$$

(229). $$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x+y) = OD = OB - EC$$
$$= OC \cos BOC = CQ \sin EQC$$
$$= \cos y \cos x - \sin y \sin x$$

(230). $$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

In the figure below, let (x) and (y) be any two acute angles each of which is less than 90° and (x) being greater than (y). QC is perpendicular to OP. BC and DQ are perpendicular to OA, and EQ is parallel to OA. The radius of the circle $(OA, OQ,$ or $OP)$ is equal to unity.



$$x - y = AOQ$$

$$\text{angle } ECQ = x;\ OC = \cos y;\ CQ = \sin y$$

$$\sin(x-y) = DQ = BC - EC$$
$$= OC \sin BOC - CQ \cos ECQ$$
$$= \cos y \sin x - \sin y \cos x$$

231). $$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x-y) = OD = OB + EQ$$
$$= OC \cos BOC + CQ \sin ECQ$$
$$= \cos y \cos x + \sin y \sin x$$

(232). $$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

Although equations (229)—(232) were derived for acute angles only, and other special requirements as noted, it may be shown that the equations are true for all values of these angles by proving the special cases separately. Throughout the foregoing derivations, it must be remembered that the lines representing the functions are the actual trigonometric functions since the radius of the circle is unity.

The derivation of the tangent of the sum and the difference of two angles is accomplished algebraically without resort to the geometric proof:

$$\text{Tan}(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$$

Divide both numerator and denominator by $\cos x \cos y$:

$$\operatorname{Tan}(x+y)=\frac{\dfrac{\sin x\cos y}{\cos x\cos y}+\dfrac{\cos x\sin y}{\cos x\cos y}}{\dfrac{\cos x\cos y}{\cos x\cos y}-\dfrac{\sin x\sin y}{\cos x\cos y}}=\frac{\dfrac{\sin x}{\cos x}=\dfrac{\sin y}{\cos y}}{1-\dfrac{\sin x\sin y}{\cos x\cos y}}$$

(233). $$\operatorname{Tan}(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

In the same way it may be shown:

(234). $$\operatorname{Tan}(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}$$

Functions of an Angle in Terms of those of Half the Angle

(235). $\sin 2x=2\sin x\cos x$

(236). $\cos 2x=\cos^2x-\sin^2x$

(237). $\cos 2x=1-2\sin^2x$

(238). $\cos 2x=2\cos^2x-1$

(239). $$\tan 2x=\frac{2\tan x}{1-\tan^2x}$$

Functions of an Angle in Terms of those of Double the Angle

(240). $2\sin^2x=1-\cos 2x$

(241). $2\cos^2x=1+\cos 2x$

(242). $$\tan^2x=\frac{1-\cos 2x}{1+\cos 2x}$$

(243). $$\tan x=\frac{\sin 2x}{1+\cos 2x}$$

(244). $$\tan x=\frac{1-\cos 2x}{\sin 2x}$$

The Product of Functions of Angles in Terms of Sums of Functions

(245). $\sin x\cos y=\tfrac12\sin(x+y)+\tfrac12\sin(x-y)$

(246). $\cos x\sin y=\tfrac12\sin(x+y)-\tfrac12\sin(x-y)$

(247). $\cos x\cos y=\tfrac12\cos(x+y)+\tfrac12\cos(x-y)$

(248). $\sin x\sin y=-\tfrac12\cos(x+y)+\tfrac12\cos(x-y)$

The Algebraic Sum of Functions of Angles in Terms of the Product of Functions

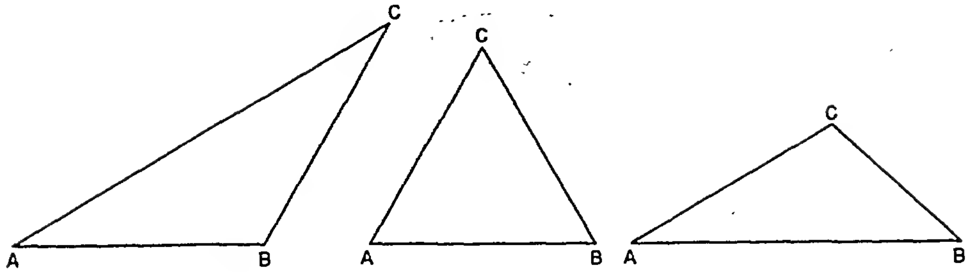
(249). $\sin x+\sin y=2\sin\tfrac12(x+y)\cos\tfrac12(x-y)$

(250). $\sin x-\sin y=2\cos\tfrac12(x+y)\sin\tfrac12(x-y)$

(251). $\cos x+\cos y=2\cos\tfrac12(x+y)\cos\tfrac12(x-y)$

(252). $\cos x-\cos y=-2\sin\tfrac12(x+y)\sin\tfrac12(x-y)$

Oblique Triangle Formulas



(215). $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$ Sine Proportion

(216). $$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ b^2 &= a^2 + c^2 - 2ac \cos B \\ c^2 &= a^2 + b^2 - 2ab \cos C \end{aligned}$$ Cosine Law

(217). $$\begin{aligned} \frac{a-b}{a+b} &= \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)} \\ \frac{a-c}{a+c} &= \frac{\tan \frac{1}{2}(A-C)}{\tan \frac{1}{2}(A+C)} \\ \frac{b-c}{b+c} &= \frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)} \end{aligned}$$ Tangent Law

(253).

$$\sin \tfrac{1}{2}A = \sqrt{\frac{(s-b)\,(s-c)}{bc}}$$

$$\sin \tfrac{1}{2}B = \sqrt{\frac{(s-a)\,(s-c)}{ac}}$$

$$\sin \tfrac{1}{2}C = \sqrt{\frac{(s-a)\,(s-b)}{ab}}$$

Half-angle formulas, where (s) is half the sum of the three sides and:

$$r = \sqrt{\frac{(s-a)\,(s-b)\,(s-c)}{s}}$$

(254).

$$\cos \tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}$$

$$\cos \tfrac{1}{2}B = \sqrt{\frac{s(s-b)}{ac}}$$

$$\cos \tfrac{1}{2}C = \sqrt{\frac{s(s-c)}{ab}}$$

(255).

$$\tan \tfrac{1}{2}A = \sqrt{\frac{(s-b)\,(s-c)}{s(s-a)}} \qquad \text{or, } \tan \tfrac{1}{2}A = \frac{r}{s-a}$$

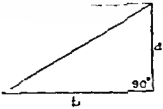
$$\tan \tfrac{1}{2}B = \sqrt{\frac{(s-a)\,(s-c)}{s(s-b)}} \qquad \text{or, } \tan \tfrac{1}{2}B = \frac{r}{s-b}$$

$$\tan \tfrac{1}{2}C = \sqrt{\frac{(s-a)\,(s-b)}{s(s-c)}} \qquad \text{or, } \tan \tfrac{1}{2}C = \frac{r}{s-c}$$

AREA OF TRIANGLES

Right Triangle

(256). The area of a right triangle is equal to one-half the product of the two sides forming the right angle.

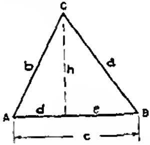


$$\text{Area} = \frac{1}{2}(a)(b) = \frac{ab}{2}$$

Oblique Triangle

(257). The area of an oblique triangle is equal to one-half the product of any two sides and the sine of an included angle.

$$\text{Area} = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B$$



$$\text{Area} = \frac{1}{2}dh + \frac{1}{2}eh = \frac{1}{2}h(d+e)$$

$$= \frac{1}{2}hc$$

$$h = b\sin A \quad \text{also, } h = a\sin B$$

$$\text{Area} = \frac{1}{2}bc\sin A$$

$$\text{Area} = \frac{1}{2}ac\sin B$$

Similarly,

$$\text{Area} = \frac{1}{2}ab\sin C$$

Another formula makes possible the finding of the area of *any* triangle when the lengths of the three sides are given.

(258).

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

Where, $s = \frac{1}{2}(a+b+c)$ $\qquad (a+b+c) = \text{perimeter}$

Section IV

LOGARITHMS

INTRODUCTION

The invention of logarithms was one of the great inventions of all times for the saving of time and labor. Many calculations which are difficult or impossible by other methods are readily made by means of logarithms. Using logarithms, the process of multiplication is replaced by one of addition; that of division, by one of subtraction; that of raising to a power, by a simple multiplication; and that of extracting a root becomes division.

Although all of the operations enumerated above can be performed using logarithms, the processes of multiplication and division are not ordinarily solved by this method as long-hand methods are simpler, more accurate, and already understood by most people. Furthermore, with the development of the modern calculating machine, a means for obtaining products and quotients has been provided which is superior to either the use of logarithms or ordinary long-hand computation for those specified operations. However, for obtaining roots and powers of numbers, the use of logarithms is not only desirable, but a necessity in many cases as most fractional roots and powers can be determined only by this means.

Before making practical use of the power and convenience of logarithms it is advisable to learn something of the principle upon which the system is based.

DEFINITIONS AND PRINCIPLES

There are four fundamental rules for operations with exponents, namely:

Multiplication, $N^a \times N^b = N^{a+b}$

or, the product of two or more like quantities (which have either like or unlike exponents) is equal to this quantity with the sum of the exponents of the factors as an exponent.

Division, $N^a \div N^b = N^{a-b}$

or, the quotient of two like quantities (which have either like or unlike exponents) is equal to this quantity with the difference between the exponents of the dividend and the divisor as an exponent.

Power, $(N^a)^b = N^{ab}$

or, any quantity raised to any power is equal to the quantity with its original exponent multiplied by the power in question.

Root, $\sqrt[b]{N^a} = (N^a)^{\frac{1}{b}} = N^{\frac{a}{b}}$

or, any quantity from which a given root is to be extracted is equal to the quantity with its exponent divided by the root in question.

If any number, such as 2, is taken as the number N referred to above, and various exponents from O to 10 are assumed, the following table can be constructed:

N	Exponent	Value
2	0	1
2	1	2
2	2	4
2	3	8
2	4	16
2	5	32
2	6	64
2	7	128
2	8	256
2	9	512
2	10	1024

This table can be used to perform the four fundamental operations with exponents by representing the numbers involved as some power of the number 2. For example:

Multiplication, $8 \times 64 = 2^3 \times 2^6 = 2^9 = 512$

Division, $\dfrac{1024}{16} = 2^{10} - 2^4 = 2^6 = 64$

Power, $16^2 = (2^4)^2 = 2^8 = 256$

Root, $\sqrt[2]{256} = \sqrt[2]{2^8} = 2^{\frac{8}{2}} = 2^4 = 16$

In the small table of the powers of 2 given above there are many gaps, because only those powers which have *whole* exponents are given For all the numbers between 16 and 32, for example, the exponents will be decimals, and will be greater than 4 but less than 5, etc In practice, the base used is not 2, but 10, and all the intermediate exponents have been computed to many decimals, these forming a table of logarithms. If 10 is taken as a base and (a) as an exponent, then, for any number,

$$N = 10^a$$

The exponent (a) is the logarithm of (N) when the base is 10, or, the *logarithm* of number is the exponent of a power to which a certain number, called the *base*, must be raised to produce the given number When the base number is not specified, it is generally understood to be 10, and the logarithms to this base are termed *common* logarithms. Thus the logarithm (common) of a number is the power to which 10 must be raised to equal the number. For common logarithms.

$$1000 = 10^3$$
$$\log 1000 = 3.000000$$
$$100 = 10^2$$
$$\log 100 = 2.000000$$
$$10 = 10^1$$
$$\log 10 = 1.000000$$

$$1 = \frac{10}{10} = \frac{10^1}{10^1} = 10^{\circ}$$

$$\log 1 = 0.000000$$

$$0.1 = \frac{1}{10} = \frac{1}{10^1} = 10^{-1}$$

$$\log 0.1 = -1.000000$$

$$0.01 = \frac{1}{100} = \frac{1}{10^2} = 10^{-2}$$

$$\log 0.01 = -2.000000$$

$$0.005 = \frac{5}{1000} = \frac{1}{200} = \frac{1}{10x} \text{(where } x > 2 \text{ but } < 3\text{)}$$

$$= \frac{1}{10^{2+c}} = 10^{-2-c} = 10^{-3+m}$$

(See below)

$$\log 0.005 = -3 + .698970 = \overline{3}.698970 \text{ (see below)}$$

$$0.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$$

$$\log 0.001 = -3.000000$$

From the foregoing, it is evident that only a very few numbers have logarithms which are integers, that is, whole numbers without any accompanying fraction. For instance, the logarithms of any number between 100 and 1000 will have a value greater than $+2$, yet less than $+3$, since $+2$ is the logariths of 100 and $+3$ is the logarithm of 1000. Such a logarithm is written, $(2+m)$ where (m) represents a positive decimal fraction called the *mantissa*. The integer portion of the logarithm, which in this case is $+2$, is termed the *characteristic*. The complete logarithm of the number is, therefore, $(2+m)$, or the sum of the characteristic and the mantissa.

The explanations and solutions of examples throughout SECTION IV are based on data obtained fom a six-place table of logarithms of numbers and a six-place table of logarithms of the trigonometric functions. However, the same procedure is employed if tables of either less or greater accuracy are employed.

N		6		
345		.538574		

The complete logarithm of a number is not obtained directly from a table of so-called logarithms because such tables are generally tabulations of mantissas only, the values of the characteristics being omitted altogether. The tables are then actual logarithms only for the numbers ranging in value from 1.0 to 9.99 inclusive, since these numbers have zero as a characteristic. (See RULES FOR CHARACTERISTICS). In finding

the mantissa part of the logarithm, no attention is given to the position of the decimal point in the given number because the mantissa is always identical for the same sequence of figures For example, the mantissa part of the logarithm of 3.456 is identical to the mantissa part of the logarithm of 345.6, the value being .538574 in both cases Rules for finding the value of the characteristic of any given number are given elsewhere If used to determine the characteristic of 345.6, the value will be found to be 2, which will verify the statement given above that the characteristic of the logarithm of any number between 100 and 1000 is 2. The complete logarithm of 345.6 is therefore 2 + 538574 or 2 538574.

The use of a so-called logarithm table for finding the mantissa part of the logarithm of a number greater than unity has already been described. The tables are similarly used in finding the mantissa part of the logarithm of a number less than unity, that is a decimal fraction However, it should be thoroughly understood that the tabulated mantissas are all positive in sign and when dealing with decimal fractions the characteristic will be negative and the mantissa positive Great care must be exercised in performing subsequent operations with such logarithms. That the characteristic will be negative and the mantissa positive for any given decimal fraction may be shown by an example. From Page 173 it is evident that the logarithm of 0 005 is a number between — 2 and — 3. This is true because the number 0 005 is in between 01 and .001 whose logarithms are — 2 and — 3 respectively Since 0 005 is smaller numerically than 01, being only one-half as large, its logarithm will be more negative than the logarithm of 01 since the value of all logarithms increase negatively as the values of the numbers become smaller The actual value of the logarithm of .005 is — 2.301030 which can also be written as — 2 000000 — 0 301030 The value of the mantissa of this logarithm (— 301030) is *not* given in the tables because the mantissas contained therein are always positive. To express the complete logarithm of .005 as a number in which the mantissa is positive it is possible to add 1 to the mantissa and subtract the same 1 from the characteristic This is obviously a valid algebraic operation since

$$\begin{aligned} -2.301030 &= -2.000000 - 0.301030 - 1 + 1 \\ &= -2.000000 - 1 - 0.301030 + 1 \\ &= -3.000000 + .698970 \\ &= \bar{3}.698970 \end{aligned}$$

The logarithm as written $\bar{3}.698970$ with the negative sign placed over the characteristic denotes that it *alone* is negative Thus, $\bar{3}.698970$ means — 3.000000 + .698970 or — 2.301030. The fallacy of writing the log 0 005 = — 3 698970 is apparent for the minus sign would indicate that both the characteristic and the mantissa are negative. Of course, the complete logarithm of a decimal fraction can be written as an entirely negative number as shown above, but then the mantissa part of the logarithm does not correspond to the mantissa as given in a table of logarithms where all mantissas are *always* positive. The facts as described above are the basis for the rule concerning the value of the characteristic of the logarithms of decimal fractions.

The complete logarithm of any number, decimal fractions or otherwise, is obtained by determining the characteristic and adding to it or annexing the value of the mantissa. The reverse order of determining the mantissa first and prefixing the characteristic is also valid since it produces the same result. The absolute value and algebraic sign of

the characteristic depends upon the position of the decimal point in the given numbers, and is easily found by either one of the two methods to follow. Mantissas are obtained directly from the table unless the given number consists of more than four digits, in which case interpolation is required, (see page 177).

RULES FOR CHARACTERISTICS

The characteristic for any number is found by either of the following methods.

Method 1.—When the given number is *greater* than one, the characteristic is positive and is one *less* than the number of places (digits) appearing to the left of the decimal point in the given number. When the given number is *less* than one, the characteristic is *negative* and is one *more* than the number of zeros between the decimal point and the first significant figure in the given number.

Method 2.—A more simple method is to remember that the characteristic is zero for any given number which has but a single place (digit) to the left of the decimal point. Moving the decimal point to the right increases the characteristic by one for each place moved. Moving the decimal point to the left decreases the characteristic (increases the negativity) by one for each place moved.

AUGMENTED LOGARITHMS

To avoid the use of negative characteristics such as $\bar{3}.301030$ it is possible to add any quantity to the logarithm provided that it is indicated that the *same* quantity is also to be subtracted.

$$\begin{aligned} \text{Log}\ .002 &= \bar{3}.301030 = -3.000000 + .301030 \\ &= -3.000000 + 10 + .301030 - 10 \\ &= +7.000000 + .301030 - 10 \\ &= 7.301030 - 10 \end{aligned}$$

The result can be obtained directly by adding a $+10$ to the characteristic, and from the resulting logarithm subtracting 10.

$$\bar{3}.301030 = 7.301030 - 10$$

The validity of this operation is further established by reducing both the given logarithm $\bar{3}.301030$ and its equivalent form $7.301030 - 10$ to their respective values as represented by single sequences of numbers.

$$\bar{3}.301030 = -3.000000 + .301030 = -2.698970$$
$$7.301030 - 10 = -10.000000 + 7.301030 = -2.698970$$

Logarithms operated on as described above are termed augmented logarithms. Other values, usually in multiples of 10, are sometimes used in place of 10. The reason for using 10 or some multiple of 10 is to eliminate errors in subtraction, because it is easier to subtract 10, 20, or 30 from any number than it is to subtract any other value. However, any number may be so employed provided that the negative characteristics of the given logarithm becomes positive and of such absolute value that when the number is subtracted from the characteristic, the original negative characteristic is regained.

$$\begin{aligned}\bar{3}.698970 &= 7.698970 - 10\\ &= 17.698970 - 20\\ &= 0.698970 - 3\\ &= 3.698970 - 6\end{aligned}$$

Another operation which cannot be strictly classified as augmenting or enlarging a logarithm is described in this section for convenience. This operation consists of changing a logarithm having a negative characteristic and a positive mantissa into an entirely negative number so that the resulting number may be used as a multiplier, or divisor For example, the logarithm of 0 002 is $\bar{3}$ 301030, which means $-3 +$ 0 301030. When written as $\bar{3}.301030$ the number cannot be used either as a multiplier or a divisor, but if the absolute value of the logarithm is determined, then it may be so used The true value of the expression $\bar{3}$ 301030 is $-$ 3.000000 + 0 301030 or,

$$\begin{array}{r} -3\,000000\\ \underline{0.301030}\\ -2\,698970\end{array}$$

These operations are not confined to logarithms having negative characteristics as implied above The same procedure may be employed to advantage in solving problems in which logarithms having positive characteristics are involved. The example below is solved by employing the previously described principles, thus avoiding, as a result of the indicated operation, a logarithm which is entirely negative.

$$\begin{aligned}\log a &= \log 24 - \log 48\\ &= 1.380211 - 1.681241\\ &= (11.380211 - 10) - 1.681241\end{aligned}$$

$$\begin{array}{r} 11.380211 - 10\\ \underline{1.681241}\\ 9.698970 - 10\end{array}$$

$$\begin{aligned}\log a &= \bar{1}.698970\\ a &= 0.50\end{aligned}$$

The reverse procedure of changing an entirely negative logarithm (both characteristic and mantissa negative) into a logarithm with a positive mantissa is required before its anti-logarithm can be found. This requirement is evident if it is remembered that all mantissas tabulated in a table of logarithmic functions are positive in sign.

$$\log x = -0.301030$$

In this case the characteristic is zero and the value of the mantissa is $-.301030$ This same value may be represented by the sum of two terms, one of which is negative and the other positive.

$$\begin{aligned}-0.301030 &= -1.000000 + .698970\\ &= \bar{1}.698970\end{aligned}$$

The result can be obtained directly by adding + 1 to the mantissa making it a positive decimal and subtracting — 1 from the characteristic. The sum of + 1 and any negative mantissa regardless of the sequence of numbers can be written down directly by

subtracting each number of the given mantissa from 9, except the right-hand digit which is subtracted from 10.

$$\operatorname{Log} x = -3.456789 = \bar{4}.543211$$

USE OF TABLES OF LOGARITHMS—INTERPOLATION

Interpolation of Mantissas

The mantissa of a number of more than four digits can be found to a high degree of accuracy by assuming that the change in the value of the mantissa is directly proportional to the change in the value of the number. This relationship actually does not exist inasmuch as it would require, for example, that the mantissa of 50 should be twice the mantissa of 25. The actual values are .698970 and .397940 respectively, which are obviously not in the ratio of 2 to 1. One exception to this reasoning might at first seem to be justified in the case of the mantissas of 4 and 2 which are .602060 and .301030 respectively. However, the value of the tabular difference for 200 is 217, and for 400 is 109, which indicates that the change in the value of the mantissa is *not* proportional to the change in value of the number.

The error resulting from the above assumption is of small consequence in any ordinary computation because it is only between the tabulated values of the mantissa that the assumption is assumed to apply, and not to the table as a whole.

The following steps are employed in finding the mantissa of a number consisting of five or more digits. It is assumed that an ordinary logarithm table is being used which allows the mantissa of a number of four or fewer digits to be read directly.

Step. 1. Find the mantissa corresponding to the first four digits of the number.

Step 2. Multiply (A) the tabular difference between the mantissa obtained in step 1 and the mantissa adjacent and next higher in the tables by (B) the fifth and following digits of the given number treated as a decimal fraction.

Step 3. Add the product obtained in step 2 to the mantissa obtained in step 1. The sum will be the desired mantissa. The number of digits retained in the interpolated value should not be greater than the number of digits occuring in the tables being used.

Mantissa 34567

N.		6	7	Diff.
345		.538574	.538699	126

Mantissa of 3456 =	538574	(Step 1.)
Mantissa of 3457 =	538699	
Mantissa of 3456 =	538574	
Tabular difference =	.000125	
(000125) (7) =	0000875	(Step 2.)
=	000087	(Either this value or the one
or, =	000088	below it may be used.)
Mantissa of 3456 =	538574	(Step 3.)
	.000088	
Mantissa of 34567 =	.538662	

Considerable effort can be spared if the tabular differences are not computed in each case as indicated in step 2 This subtraction is unnecessary if the difference column to the right of each page of the tables is employed. This column indicates the *average difference*, with no regard to decimal point, between the mantissas of any two columns on the same line This tabulated average difference corresponds to the exact difference in most cases where large numbers are involved and the value of the mantissa is changing slowly For small values the last digit of the tabulated difference should be checked against the actual difference of the last digits of the two mantissas if the maximum accuracy is desired In the specific example just solved the average tabular difference is shown as 126, while the actual tabular difference is 125.

Interpolation in Finding Numbers

After a logarithm or number of logarithms have been operated on as described in the paragraphs entitled FUNDAMENTAL OPERATIONS USING LOGARITHMS, the next and final operation is to find the numerical value of the answer which is represented by its logarithm Such a number is termed the anti-logarithm of the logarithm, or simply anti-log. The anti-log is obviously the answer in ordinary numbers, and is found by process just the reverse of finding the logarithm

In finding the logarithm of a number the process of interpolation is required in only a fraction of the total number of times because numbers of four or fewer digits are most frequently involved Unfortunately, in finding anti-logarithms the need for interpolation is the rule rather than the exception because the given mantissa is rarely found to correspond to a mantissa included in the tables. Where interpolation is necessary the procedure is exactly a reverse process of that previously described for finding mantissae This procedure is most easily explained by an example

$$\text{Log}\, x = 337110$$
$$x =$$

N		3	4		Diff.
217		.337060	.337260		200

Step 1. From the tables it is evident that the given mantissa has a value which lies in between .337060 and .337260. These values of mantissa correspond to the sequence of numbers 2173 and 2174 respectively. The sequence of numbers corresponding to a mantissa of .337110 is therefore 2173 +.

Step 2. The given mantissa exceeds the mantissa of 2173 by .000050.

Mantissa of given number	=.337110
Mantissa corresponding to 2173	=.337060
Difference	.000050

And, the mantissa of 2174 exceeds the mantissa of 2173 by 200.

Mantissa corresponding to 2174	=.337260
Mantissa corresponding to 2173	=.337060
Tabular difference	=.000200

Assuming that the change in the value of the mantissa is directly proportional to the change in the value of the number, the fifth and succeeding numbers of the sequence 2173 can be determined by the ratio whose numerator is the amount by which the value of the given mantissa exceeds the mantissa of 2173, and whose denominator is the total tabular difference of the mantissas of 2173 and 2174 respectively.

$$\frac{50}{200} = .25$$

Step 3. The sequence of figures is therefore:

217325

Since the given logarithm had a characteristic of zero,

$$x = 2.17325$$

The interpolation fraction is rarely of such magnitude that division can be performed exactly. In most cases the quotient will be an irrational number allowing no exact solution. For any ordinary problem the results obtained by performing this division with a slide rule are satisfactory. (See page 205).

FUNDAMENTAL OPERATIONS USING LOGARITHMS

The common logarithm of a number is the power to which the number 10 must be raised to equal the given number. Therefore, since logarithms are exponents, the operations involving logarithms are governed by the same rules that apply to exponents. As stated on the first page of this section, logarithms may be used in all computations except addition and subtraction. The operations of multiplication and division are more easily performed by other methods, but for the operations involving roots and powers of most numbers the use of logarithms is desirable, and, in a great number of cases, an absolute necessity.

The detailed description which has been given in the previous paragraphs may make it appear that the amount of labor involved in the use of logarithms is considerable. Actual application, however, will demonstrate the brevity and simplicity of the system, especially if exact interpolation is not required.

Multiplication

$$(10^a)\,(10^b) = 10^{a+b} \qquad \text{(Rule 29)}$$

(259) The logarithm of the product of two or more quantities is equal to the sum of the logarithms of the factors being multiplied together. The anti-logarithm of this sum is the product of the several factors.

$$\begin{aligned}
x &= (a)\,(b)\\
\log x &= \log(a) + \log(b)\\
x &= \text{anti-log}[\log(a) + \log(b)]\\
x &= (391)\,(375)\\
\log x &= \log 391 + \log 375\\
&= 2.592177 + 2.574031\\
\log x &= 5.166208\\
x &= 146625.00
\end{aligned}$$

Division

$$10^a \div 10^b = 10^{a-b} \qquad \text{(Rule 31)}$$

(260) The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor. The anti-logarithm of this difference is the quotient.

$$\begin{aligned}
x &= a/b\\
\log x &= \log(a) - \log(b)\\
x &= \text{anti-log}[\log(a) - \log(b)]\\
x &= \frac{48}{24}\\
\log x &= \log 48 - \log 24\\
&= 1.681241 - 1.380211\\
\log x &= 0.301030\\
x &= 2.00
\end{aligned}$$

Powers

$$(10^a)^n = 10^{an} \qquad \text{(Rule 34)}$$

(261). The logarithm of any power of any number is equal to the product of the power (exponent) and the logarithm of the number. The anti-logarithm of this product is the value of the number raised to the power in question.

$$\begin{aligned}
x &= a^n\\
\log x &= n \log a\\
x &= \text{antilog}\,(n \log a)\\
x &= 25^3\\
\log x &= 3 \log 25 = 3(1.397940)\\
\log x &= 4.193820\\
x &= 15625.00
\end{aligned}$$

The multiplication of (3) (1.397940) in the solution above could have been solved by logarithms, and in problems where both factors are of many digits such an operation would result in a great saving of time. Using the above example:

$$\log x = (3)\ (1.397940)$$
$$\log(\log x) = \log 3 + \log 1.397940$$
$$= 0.477121 + 0.145488 = 0.622609$$
$$\log x = 4.193816$$
$$x = 15624.86$$

Roots

$$\sqrt[b]{10^a} = (10^a)^{1/b} = 10^{a/b} \qquad \text{(Rule 35)}$$

(262). The logarithm of any root of any number is equal to the logarithm of the number divided by the root in question. The antilogarithm of this quotient is the value of the required root.

$$x = \sqrt[n]{a}$$

$$\log x = \frac{1}{n}\log a \ \text{ or } \ \frac{\log a}{n}$$

$$x = \sqrt{625}$$

$$\log x = \frac{1}{2}\log 625 = \frac{\log 625}{2} = \frac{2.795880}{2}$$

$$\log x = 1.397940$$
$$x = 25.000\ldots$$

The four basic operations which can be performed by logarithms are all demonstrated in the previous paragraphs. These procedures must be thoroughly understood before attempting the same operations with less simple numbers, and in performing several operations all within one equation.

COLOGARITHMS

It has been shown in Section 1, (Rule 176) that the quotient obtained when (a) is divided by (b) is the same as the product of (a) and the reciprocal of (b), or:

$$\left(\frac{a}{b}\right) = \frac{1}{b}(a)$$

When this principle is applied to logarithmic operation it is possible to perform an indicated division by adding the logarithm of the dividend (numerator) to the logarithm of the reciprocal of the divisor (denominator).

$$\log\left(\frac{a}{b}\right) = \log\left(\frac{1}{b}\right) + \log a$$

The logarithm of the reciprocal of any number such as (b) is $\log\ (1/b)$ and is numerically the same as $-\log b$.

$$\log\left(\frac{1}{b}\right) = \log 1 - \log b$$

but

$$\log 1 = 0$$

$$\log\left(\frac{1}{b}\right) = 0 - \log b = -\log b$$

The logarithm of the reciprocal of a number has been given the specific title of cologarithm. This title is appropriate since in converting a cologarithm (which is the negative logarithm of the number) to a logarithm having a positive mantissa, the given mantissa is added to 1 which gives a positive decimal whose absolute value is the complement of the original negative characteristic. Since $+1$ has been added to the mantissa of the logarithm, it is necessary to subtract 1 from the characteristic. These operations have already been described in detail on page 175.

$$x = \frac{1}{12}$$

$$\log x = \log\left(\frac{1}{12}\right) = \text{colog } 12 = -\log 12$$

DIVISION OR MULTIPLICATION OF LOGARITHMS

There are some special cases in which one logarithm is to be either multiplied or divided by another logarithm. Such operations are not very frequent and one rarely becomes well enough acquainted with this type of problem to be at all certain of the validity of the results obtained. In many instances the answer will not be correct because of an erroneous procedure, specifically, the addition of logarithms instead of multiplication, or the subtraction of logarithms instead of division. This fallacy may be partially eliminated by writing the solution of the problem in algebraic form which is a recommended procedure in any solution, and then following out the operations indicated by the equation.

$$2^x = 25$$
$$x \log 2 = \log 25$$
$$(x)\ (0.301030) = 1.397940$$
$$x = \frac{1.397940}{.301030} = 4.6438$$

Note that x does not equal $1.397940 - .301030$

The equation says $\dfrac{1.397940}{.301030}$ (Rule 40)

The result is apparently correct since $2^4 = 16$ and $2^5 = 32$. In dividing the two logarithms to obtain the quotient, 4.6438, it would have been possible to employ logarithms inasmuch as this would be equivalent to the example shown on page 180. However, actual division seems to be the more practical solution in this case.

$$\frac{.1875}{.750} = \left(\frac{24}{48}\right)^x$$

This problem can be solved by inspection as the value of (x) is obviously 2. However, such convenient problems do not exist in actual practice. The solution of the problem assumes that the given ratios are of such value that they cannot be simplified other than by the operations shown.

$$\frac{750}{187.5}=\left(\frac{48}{24}\right)^x \qquad \text{(Rules 9, 41)}$$

$$\log 750-\log 187.5=x(\log 48-\log 24)$$

$$x=\frac{\log 750-\log 187.5}{\log 48-\log 24}=\frac{2.875061-2.273001}{1.681241-1.380211}$$

$$x=\frac{0.602060}{0.301030}=2$$

Note that $\log 48-\log 24$ does not equal $\log(48-24)$. The difference in the logs of two numbers is not equal to the log of the difference of the two numbers. This is often erroneously assumed to be true.

SOLUTION OF EQUATIONS USING LOGARITHMS

Complex equations involving several operations are solved by proper applications of Rules 259, 260, 261, and 262. The writing of the solutions in algebraic forms is recommended in all cases.

$$\begin{aligned} 133.5 &= P\,(6)^{1.3} \\ \log 133.5 &= \log P + 1.3 \log 6 \\ 2.125481 &= \log P + 1.3\,(0.778151) \\ 2.125481 - 1.011596 &= \text{Log}\, P \\ \log P &= 1.113885 \\ P &= 12.9983 \end{aligned} \qquad \boxed{\frac{276}{334}}$$

NOTE: The fraction appearing within the rectangular square is the interpolation fraction used in obtaining the final result. This information is not always shown as a part of the solution since the actual interpolation is often performed by the use of a slide rule or a table of proportional parts. The interpolation fraction is included in this and the following problems so that these solutions may serve as additional examples to those already given in the paragraphs entitled Interpolation in Finding Numbers.

$$D=300\sqrt[4]{\frac{550}{(2200)^2\,(182)}}$$

$$\begin{aligned} \log D &= \log 300 + 1/4(\log 500 - 2\log 2200 - \log 182) \\ &= 2.477121 + 1/4\,[2.740363 - 2(3.343423) - 2.260071] \\ &= 2.477121 + 1/4\,(2.740363 - 6.684846 - 2.260071) \\ &= 2.477121 - \frac{6.204554}{4} = 2.477121 - 1.551139 \\ &= .925982 \end{aligned}$$

$$D = 8.4330$$

$$175 = (K)\,(2375')$$

$$150 = (K)\,(2100')$$

$$\begin{array}{l} \log 175 = \log K + x \log 2375 \\ \underline{\log 150 = \log K + x \log 2100} \\ \log 175 - \log 150 = x\,(\log 2375 - \log 2100) \end{array}$$

$$x = \frac{\log 175 - \log 150}{\log 2375 - \log 2100} = \frac{2.243038 - 2.176091}{3.375664 - 3.322219}$$

$$x = \frac{0.066947}{0.053445} = 1.2526335$$

$$\begin{aligned} x &= 1.25 \text{ (approx.)} \\ \log 175 &= \log K + 1.25\,(\log 2375) \\ 2.243038 &= \log K + 1.25\,(3.375664) \\ \log K &= 2.243038 - 4.219580 = -1.976542 \\ \log K &= \bar{2}.023458 \\ K &= 0.010555 \end{aligned} \qquad \boxed{\begin{array}{c} 206 \\ \hline 412 \end{array}}$$

Problems containing numbers with fractional exponents such as a/b may be solved by using the exponent either in the given form (a/b), or by using its decimal equivalent. Since the exact decimal equivalent of many fractions can not be expressed, even though the division is carried to several places, the results obtained using the decimal form will be slightly less accurate than if the original fractional form had been retained.

Example: $$y = \left(\frac{0.08726}{0.1321}\right)^{5/3}$$

$$\log y = 5/3 \log \left(\frac{0.08726}{0.1321}\right) = 5/3\,(\log 0.08726 - \log .1321)$$

$$\begin{aligned} \log y &= 5/3\,(\bar{2}.940815) - (\bar{1}.120903) \\ &= 5/3\,(18.940815 - 20) - (9.120903 - 10) \\ &= 5/3\,(9.819912 - 10) = 5/3\,(29.819912 - 30) \\ &= 5\left(\frac{29.819912 - 30}{3}\right) = 5\,(9.939971 - 10) \\ &= 49.699855 - 50 = \bar{1}.699855 \\ y &= 0.50101979 \\ &= .50102 \text{ (approx.)} \end{aligned} \qquad \boxed{\begin{array}{c} 17 \\ \hline 86 \end{array}}$$

Logarithms are not applicable to the operations of addition or subtraction, yet in some equations the various terms, although added or subtracted, are in themselves, products, quotients, powers or roots. The solution of such equations may be written out algebraically by the insertion of the word "antilog" in the proper place as shown below. Or it may be advisable to remove such terms from the equation, compute its value by the use of logarithms, and then replace the given term by its numerical equivalent in the equation. The resulting equation can then be solved by ordinary algebraic methods.

$$E = \left[1 - \frac{1}{6.4}^{.408}\right]100$$

$$\begin{aligned}
\log E &= \log\,[1 - \text{antilog}\,.408\,(-\text{Log}\,6.4)] + \log 100\\
\log E &= \log\,[1 - \text{antilog}\,.408\,(-0.806180)] + \log 100\\
\log E &= \log\,[1 - \text{antilog}\,(-.328921)] + \log 100\\
\log E &= \log\,[1 - \text{antilog}\,1.671079)] + \log 100\\
\log E &= \log\,[1 - .4689] + \log 100\\
\log E &= \log .5311 + \log 100 = 1.725176 + 2.00000\\
\log E &= 1.725176\\
E &= 53.11
\end{aligned}$$

An alternate solution appears to be considerably simpler.

$$\begin{aligned}
\text{let } Y &= \left(\frac{1}{6.4}\right).408\\
\log Y &= .408\,(-\log 6.4) = .408\,(-0.806180)\\
\log Y &= -.328921 = 1.617079\\
Y &= .4689\\
E &= (1 - .4689)100 = (.5311)100 = 53.11
\end{aligned}$$

SOLUTION OF TRIANGLES USING LOGARITHMS

The arithmetical labor required in the solution of triangles is reduced if the multiplication and division of the sides and functions of the angles are replaced by the addition and subtraction of the logarithms of the numbers involved. The multiplication and division of numbers by the use of logarithms is described in the section entitled FUNDAMENTAL OPERATIONS WITH LOGARITHMS.

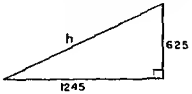
The logarithms of any trigonometric function of a given angle can be determined by first finding the value of the trigonometric function from a table of natural trigonometric functions, and then finding the logarithm of this number from an ordinary table of logarithms. The angle corresponding to a given logarithm of a trigonometric function can be determined by the reverse to this procedure.

However, tables are available which give directly the value of the logarithms of the functions of all angles. These tables are designated as tables of logarithmic functions to differentiate them from tables of natural trigonometric functions. Whenever such a table is available it should be used, since the solution of the problem is greatly simplified.

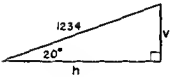
The solution of triangles by logarithms involves the same principles and formulas that are used in solutions by natural functions. These formulas are simply the definitions of the trigonometric functions, sine, cosine, tangent, etc., in the case of right triangles, or the sine proportion and the tangent law when applied to oblique triangles. Since the cosine law is not expressed in terms of either ratios or products, it is not adapted to use with logarithms. For this reason the law of tangents has been included in the section entitled OBLIQUE TRIANGLES SOLVED BY SPECIAL FORMULAS. This formula is adapted to logarithmic computations and may be used to solve one of

the types of oblique triangles ordinarily solved by the cosine law, that is, when two sides and the included angle are given The remaining case covered by the cosine law, three sides only, may be solved by logarithms by using a set of formulas known as the half-angles which are tabulated on page 169

Logarithmic Solution of Right Triangles



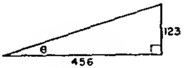
$$h^2 = 625^2 + 1245^2$$
$$h^2 = \text{antilog}\,(2 \log 625) + \text{antilog}\,(2 \log 1245)$$
$$h^2 = \text{antilog}\,(2 \times 2.795880) + \text{antilog}\,(2 \times 3.095169)$$
$$h^2 = \text{antilog}\,5\,591760 + \text{antilog}\,6\,190338$$
$$h^2 = 390{,}525 + 1{,}555{,}024 = 1{,}945{,}529$$
$$\log h^2 = 2 \log h = \log 1{,}945{,}529 = 6\,289{,}038$$
$$\log h = 3.144{,}519$$
$$h = 1394\,82$$



$$v = 1234 \sin 20°$$
$$\log v = \log 1234 + \log \sin 20°$$
$$\log v = 3.091315 + (9\,534052 - 10)$$
$$\log v = 2\,625367$$
$$v = 422.05$$

$$h = 1234 \cos 20°$$
$$\log h = \log 1234 + \log \cos 20°$$
$$\log h = 3\,091315 + (9.972986 - 10)$$
$$\log h = 3\,064301$$
$$h = 1159.58$$

In finding log sin 20° from a table of logarithmic functions, the value is found to be 9.534052. The (— 10) is not annexed to the tables in favor of brevity, but must be used in the solution. That the characteristic of the log sin 20° is negative is evident from the fact that the natural sin of any angle is always less than unity. The use of the (— 10) in conjunction with log cos 20° is governed by the same principle.



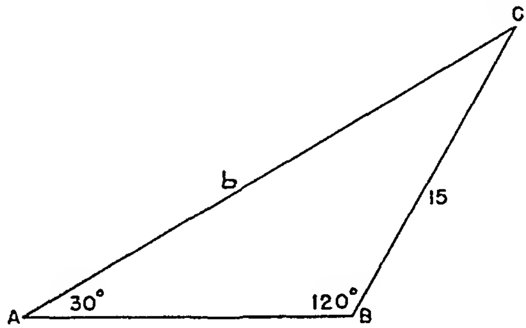
$$\tan \theta = \frac{123}{456}$$

$$\log \tan \theta = \log 123 - \log 456$$
$$\log \tan \theta = 2.089905 - 2\,658965$$
$$\log \tan \theta = (12.089905 - 10) - 2.658965$$
$$\log \tan \theta = 9\,430940 - 10$$
$$\theta = 15°4.73'$$

Logarithmic Solution of Oblique Triangles

Two Angles and Any Side

(This is equivalent to having all three angles and any side given.)

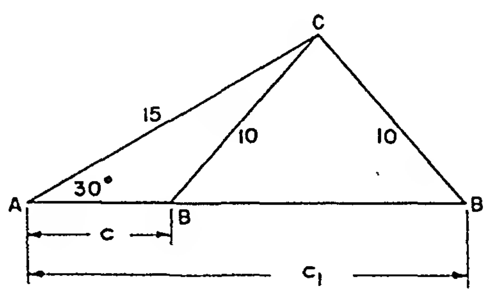


$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$b = \frac{(a)\ (\sin B)}{\sin A} = \frac{(15)\ (\sin 120°)}{\sin 30°}$$

$\log b = \log 15 + \log \sin 120° - \log \sin 30°$
$\log b = 1.176091 + (9.937531 - 10) - (9.698970 - 10) = 1.41465$
$b = 25.98$ (approximately)
$C = 180° - 30° - 120° = 30°$
$c = 15$

Two Sides and an Angle Opposite One of Them—Ambiguous Case



$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\sin B = \frac{(b)\ (\sin A)}{a} = \frac{(15)\ (\sin 30°)}{10}$$

$\log \sin B = \log 15 + \log \sin 30° - \log 10$
$\log \sin B = 1.176091 + (9.698970 - 10) - 1.00000$
$\log \sin B = 9.875061 - 1$
$B = 48°\ 35.42'$ or $180° - 48°\ 35.42' = 131°\ 24.58'$
$C = 180° - 30° - 48°\ 35.41' = 101°\ 24.58'$ or,
$180° - 30° - 131°\ 24.58 = 18°\ 35.42'$

$$\frac{a}{\sin A} = \frac{c_1}{\sin 18^\circ\, 35.42'}$$

$$c_1 = \frac{(10)\ (\sin 18^\circ\, 35.42')}{\sin 30^\circ}$$

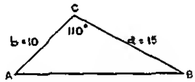
$$\begin{aligned}
\log c_1 &= \log 10 + \log \sin 18^\circ\, 35.42' - \log \sin 30^\circ \\
\log c_1 &= 1.000000 + (9.503518 - 10) - (9.698970 - 10) \\
\log c_1 &= 0.804548 \\
c_1 &= 6.38
\end{aligned}$$

$$\frac{a}{\sin A} = \frac{c}{\sin 101^\circ\, 24\, 58'}$$

$$c = \frac{(10)\ (\sin 101^\circ\, 24\, 58')}{\sin 30^\circ}$$

$$\begin{aligned}
\log c &= \log 10^\circ + \log \sin 101^\circ\, 24\, 58 - \log \sin 30^\circ \\
\log c &= 1.000000 + (9.991332 - 10) - (9.698970 - 10) \\
\log c &= 1.292362 \\
c &= 19.60
\end{aligned}$$

Two Sides and the Included Angle



$$\frac{\tan \tfrac{1}{2}(A - B)}{\tan \tfrac{1}{2}(A + B)} = \frac{a - b}{a + b}$$

$$A = \tfrac{1}{2}(A + B) + \tfrac{1}{2}(A - B) \qquad \text{(Rule 227)}$$

$$B = \tfrac{1}{2}(A + B) - \tfrac{1}{2}(A - B) \qquad \text{(Rule 228)}$$

$$A + B = 180^\circ - 110^\circ = 70^\circ$$

$$\tan \tfrac{1}{2}(A - B) = \frac{\tan \tfrac{1}{2}(A + B)\ (a - b)}{(a + b)}$$

$$\tan \tfrac{1}{2}(A - B) = \frac{\tan \tfrac{1}{2}(70^\circ)\ (15 - 10)}{(15 + 10)} = \frac{(\tan 35^\circ)\ (15)}{25}$$

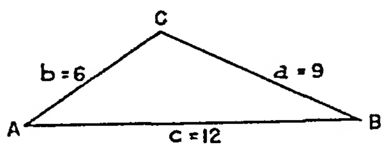
$$\begin{aligned}
\log \tan \tfrac{1}{2}(A - B) &= \log \tan 35^\circ + \log 5 - \log 25 \\
&= (9.845277 - 10) - 0.698970 - 1.397940 = 9.146307 - 10 \\
\tfrac{1}{2}(A - B) &= 7^\circ\, 58.37' \\
A - B &= 15^\circ\, 56.74'
\end{aligned}$$

$$\tan \tfrac{1}{2}(A + B) = \frac{\tan \tfrac{1}{2}(A - B)\ (a + b)}{(a - b)}$$

$$\tan \tfrac{1}{2}(A + B) = \frac{\tan \tfrac{1}{2}(15^\circ\, 56.74')\ (15 + 10)}{(15 - 10)} = \frac{(\tan 7^\circ\, 58.37')\ (25)}{5}$$

$$\log \tan \tfrac{1}{2}(A+B) = \log \tan 7^\circ 58.37' + \log 25 - \log 5$$
$$= (9.146307 - 10) + 1.397940 - .698970 = 9.845277 - 10$$
$$\tfrac{1}{2}(A+B) = 35^\circ 0.19'$$
$$A+B = 70^\circ 0.38'$$
$$A = \tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B) = 35^\circ 0.19 + 7^\circ 58.37' = 42^\circ 58.56'$$
$$B = \tfrac{1}{2}(A+B) - \tfrac{1}{2}(A-B) = 35^\circ 0.19 - 7^\circ 58.37' = 27^\circ 1.82'$$

Three Sides Given



$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)\,(s-c)}{bc}} \qquad \text{(Rule 253)}$$

$$s = \tfrac{1}{2}(a+b+c) = \tfrac{1}{2}(9+6+12) = 13.5$$
$$(s-b) = 13.5 - 6 = 7.5$$
$$(s-c) = 13.5 - 12 = 1.5$$

$$\sin\frac{A}{2} = \sqrt{\frac{7.5 \times 1.5}{6 \times 12}}$$

$$\log\sin\frac{A}{2} = \tfrac{1}{2}(\log 7.5 + \log 1.5 - \log 6 - \log 12)$$

$$= \tfrac{1}{2}(0.875061 + 0.176091 - 0.778151 - 1.079181)$$

$$\log\sin\frac{A}{2} = \tfrac{1}{2}(-0.806180) = -0.403090 = \bar{1}.596910$$

$$\frac{A}{2} = 23^\circ 17.02'$$

$$A = 46^\circ 34.04'$$

NATURAL OR NAPERIAN LOGARITHMS

Although logarithms to the base 10 are most commonly employed, there are several instances in which another base is used. This base is an irrational number which for all practical purposes is taken as 2.718. For simplicity, this value is referred to as (e), and logarithms to this base are called logarithms to the base (e) (written $\log_e$). These logarithms are also designated as either natural or Naperian logarithms to distinguish them from common logarithms.

Tables of natural logarithms which are used as described below are available. However, the $\log_e$ of any number can be found using a table of common logarithms together with the appropriate conversion factors. These factors are irrational numbers 2.302585 . . . and its reciprocal 0.434294 . . . . For simplicity, these are assumed to be 2.3026 and 0.4343 respectively.

$$\log_e A = 2.3026 \log_{10} A \quad \text{(where } A \text{ is any number)}$$
$$\log_e A = 0.4343 \log_e A$$

It is apparent that the natural logarithm of any number is approximately equal to 2.3026 times the common logarithm of that number, and conversely, the common logarithm is 1/2.3026 or 0.4343 times the natural logarithm. The natural logarithms of the numbers 1 to 10 are listed in all natural logarithm tables. Some tables include fractional numbers (0.1 to 1.0) as well. The method of obtaining natural logarithms as now described assumes the use of the less extensive table. For the logarithms of numbers 1 to 10 the values are given directly in the tables. Thus

$$\log_e 2 = 0.6931$$
$$\log_e 4 = 1.3863$$
$$\log_e 8 = 2.0794$$

Where interpolation is required, the same procedure is employed as for common logarithms.

To obtain $\log_e$ of any number less than 1.0 or greater than 10 it should be noted that when the decimal point of the number is moved one place to the right the natural logarithm of the resulting number can be obtained by adding 2.3026 to the natural logarithm of the original number. Likewise 2.3026 should be subtracted when the decimal point is moved one place to the left. Moving the decimal point (n) places in either direction requires the addition or subtraction of n times 2.3026 to the natural logarithm of the original number. These facts may be used to formulate rules for finding the natural logarithms of numbers not given directly in the tables.

Step 1. Move the decimal point in the original number so that the number produced has an absolute value between 1 and 10.

Step 2. Find the natural logarithm of the number obtained in step 1.

Step 3. If the decimal point in step 1 was moved n places to the left, the logarithm obtained in step 2 should be *increased* by the product of $n(2.3026)$.

$$\log_e 125 = \log_e 1.25 + 2(2.3026)$$
$$= 0.2231 + 4.6052 = 4.8283$$

If the decimal point in step 1 was moved n places to the right, the logarithm obtained in step 2 should be *decreased* by the product of $n(2.3026)$.

Section V

ANALYTICAL GEOMETRY OF STRAIGHT LINES

INTRODUCTION

The chief feature of analytical geometry, which distinguishes it from ordinary plane geometry, is the extensive use of algebraic and trigonometric methods. The explanation of the principles of this subject is made by the use of graphs or curves of the various equations. For simplicity, and because they are of most practical importance, the analytical geometry of straight lines only is considered in this section.

To graphically represent an equation involving two variables, it is necessary to establish a number of points each of which will have coordinates (x) and (y) which will satisfy the given equation. The numerical values for the coordinates of any one point are found by assigning some arbitrary numerical value to one of the variables and then solving the equation for the corresponding value of the other variable. A number of such points can be similarly established and through these a continuous curve or line can be drawn to represent the equation.

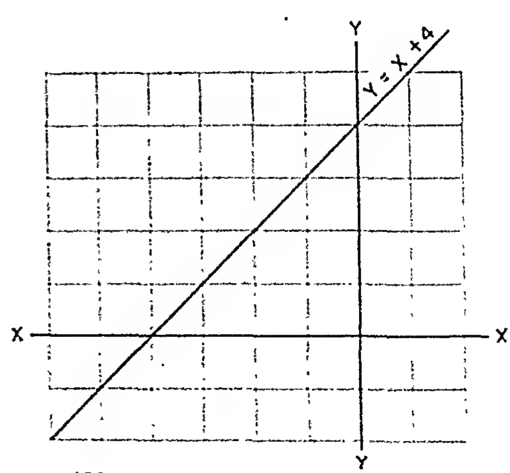
The variable to which the arbitrary value is assigned is called the *independent variable,* and the remaining unknown, since it is a function of the independent variable, becomes the *dependent variable.* The value of the independent variable is customarily plotted as abscissa (x) and the value of the dependent variable as ordinate (y). However, this order may be reversed at any time, even when plotting successive points in the same equation.

The plotting of a typical equation is shown by the following example.

$$y = x + 4$$

This equation may be rearranged in any manner which may simplify the finding of pairs of value for the coordinates (x) and (y), but it is entirely workable in its given form. The values of various pairs of the independent and dependent variables are found and tabulated as shown.

LET X =	THEN Y =
0	4
2	6
−2	2
−4	0
−6	−2

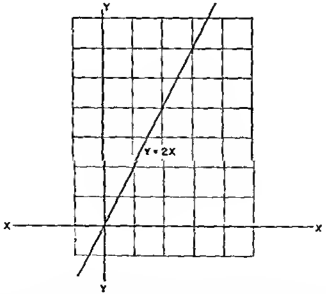


These points, when plotted indicate that the given equation might represent a straight line. Inasmuch as each of the several values assigned to the independent variable was chosen at random, it seems unnecessary to plot additional points, but as an expedient to draw a straight line through those points already established. This line, called the curve of the equation, will indicate at a glance what value of (y) corresponds to any given value of (x), or vice-versa.

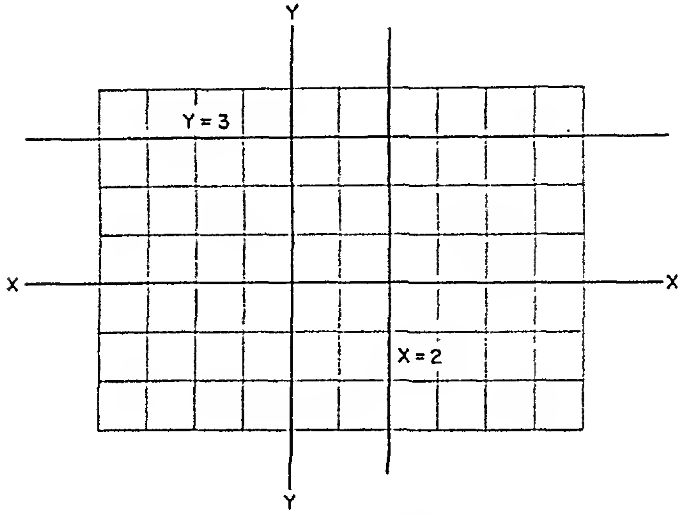
STRAIGHT LINES

An equation of the general form $Ax \pm By \pm C = 0$, is called a linear equation since all its plotted points will fall on a straight line This will always be the case where both of the variables appear separately, that is, not as their product, xy, or their quotient, x/y, and with exponents equal to one Since a straight line is determined by knowing two points on it, the graph of a linear equation can be drawn when only two points have been plotted Usually, but not necessarily always, the most convenient points to choose are those located where the line crosses the two axes These two points are found by assuming $x = 0$ and finding (y), and then letting $y = 0$ and finding (x). The two values thus found for (x) and (y) are called *intercepts* as the line crosses the axes at these points

In some cases the X-intercept and the Y-intercept are both zero and consequently the line passes through the origin. This condition is at once apparent whenever the value of the dependent variable is zero for a corresponding zero value of the independent variable Straight lines which pass through the origin can be plotted whenever one other pair of values of (x) and (y) are known.

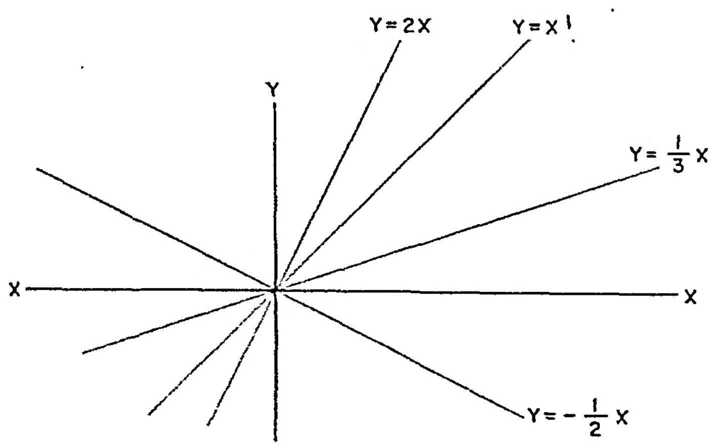


In some cases only one variable is represented in the equation, as for example $x = 2$, $y = 3$, etc. In these examples, the curve of the equation is a straight line parallel to one of the reference axes. When $x = a$, the graph is a straight line parallel to the Y-Y axis; and $y = b$ is a straight line parallel to the X-X axis.



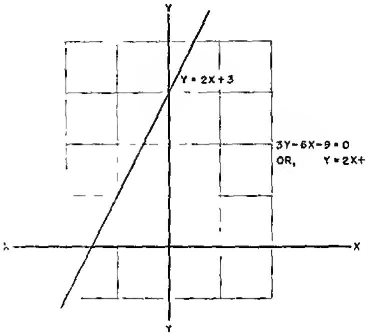
Slope of a Straight Line

As a point moves along a straight line, any change in (x) produces a definite change in (y). In the equation $y = x$ it is apparent that for any change in (x) the value of (y) will be equally affected. This ratio of the *change* in (y) to the *change* in (x) is called the slope of the curve, and for the equation $y = x$ its numerical value is equal to one. In the equation $y = 2x$ any change in (x) causes twice as large a change in (y), that is, it has a slope of two. The slope of a line is considered positive when it ascends from left to right; and a line which descends from left to right has a negative slope. A line parallel to the X-X axis has a zero slope, and one perpendicular to the X-X axis is said to have no slope.



An inspection of the lines so far plotted show that where the slope is positive, (y) increases as (x) increases, and where the slope is negative, (y) decreases as (x) increases. Furthermore, where the slope is more than one, the value of (y) changes at a greater rate than (x); where the slope is less than one, the value of (y) changes at a

rate smaller than (x) It is apparent that the slope term is useful in indicating the direction in which the curve of the equation will exist when plotted. This slope term will always be the coefficient of the (x) term when the given equation is solved for (y). For example, the slope of the equatinn $3y-6x-9=0$, is 2 since this is the coefficient of the (x) term when the equation is solved for (y), as shown below.

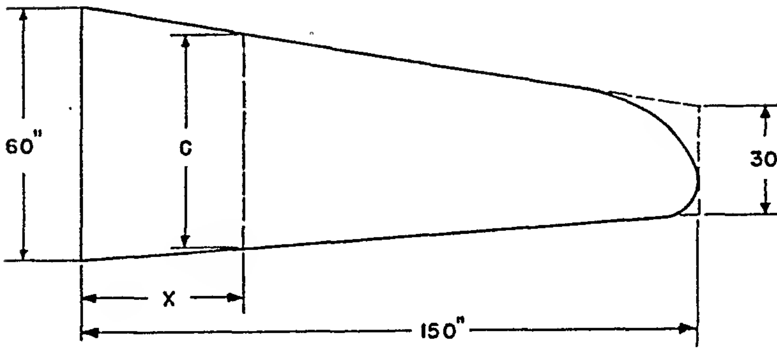


Any constant term, such as 3 in the above example, which remains after the equation is solved for (y), will signify that the line does not pass through the origin. Instead, the line crosses the Y-Y axis at the ordinate corresponding to the numerical value of the constant term (customarily represented by b). The general equation for a straight line is now written:

$$(263). \qquad y = mx + b$$

In this equation, (m) and (b) are the slope and Y-intercept respectively These quantities may be integers or fractions, and of either plus or minus value This form of the equation for a straight line is known as the slope-intercept form, since it indicates at a glance what the slope and Y-intercept of any given equation might be. If in any case there is no constant, or (b) term, then the equation becomes $y = mx$, and as shown in Fig. 0 the line passes through the origin.

There are many practical applications of the straight line equation when written in the slope-intercept form, especially in problems relating to motion, energy, lead diagrams, etc. The following example demonstrates its use in finding the chord length of an airplane wing when the chord length varies inversely as to its distance from the center of the airplane.



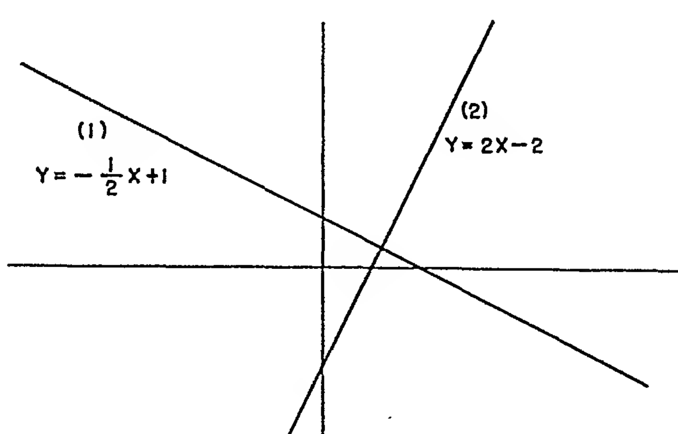
$$C = 60 - \left[\frac{60 - 30}{150}\right] x = 60 - .2x$$

Interpretation of Slope Term

As previously stated, a positive slope indicates that y *increases* with x, and a negative slope indicates that y *decreases* with x. Where (m) is greater than one, the value of (y) changes at a greater rate than (x). Where (m) is less than unity the value of (y) changes at a smaller rate than (x).

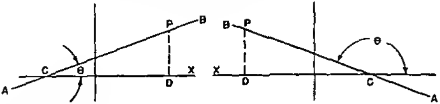
It is apparent that equations having equal slopes will plot as parallel lines. Two lines which intersect at right angles may be identified without plotting by noting that the slope of one is the *negative* reciprocal of the other. One number is the *negative* reciprocal of another if their product is -1. Thus, the negative reciprocal of 3 is $-1/3$; of $-2/5$ is $5/2$; etc. Obviously the reciprocal of a fraction is merely the fraction inverted.

$$(A) \qquad 2x + 4y = 4$$
$$y = -1/2(x) + 1$$
$$(B) \qquad -2x + y = -2$$
$$y = 2x - 2$$



The slope term (m) of a straight line may be used to advantage in engineering problems. So far the slope has been defined as the ratio of the change of ordinate (y)

to the change in abscissa (x) between any two points on the line. The slope of a straight line may also be defined as the tangent of the angle between the axis and the given line. The straight line AB cuts the X-X axis at point C. There are altogether four angles formed, but θ is the only angle commonly considered, it being the angle between the initial CX and the terminal side CB. At any point on the terminal side such as P, a line is drawn perpendicular to the X-X axis.



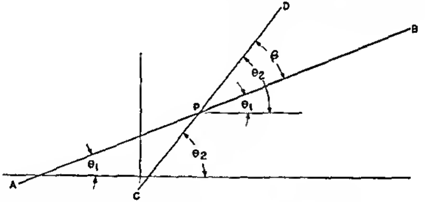
From trigonometry,

$$\tan\theta = PD/CD$$

But PD/CD is the slope of the line by fundamental definition, since it is the ratio of the change in ordinate to the change in abscissa between points C and P.
Therefore,

$$\tan\theta = m.$$

Where two straight lines intersect, and the angle between them is desired, it is possible to arrive at a solution by first finding the difference in slopes of the two lines.



Let straight lines AB and CD intersect at point P. The angle formed is β or,

$$\beta = \theta_2 - \theta_1$$

Therefore $\quad \text{Tan}\,\beta = \text{Tan}\,(\theta_2 - \theta_1)$
From trigonometry, (Rule 234)

$$\tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}$$

But $\text{Tan}\,\theta_2 = m_2$, and $\text{Tan}\,\theta_1 = m_1$ where m_1 is the slope of AB.

(264). $$\text{Tan}\,\beta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

If θ_2 is always taken greater than θ_1, then Tan β will be positive when β is less than 90° and negative when β is greater than 90°.

Example: Find the angle between the two straight lines whose equations are

$$(A)\quad x+2y+2=0$$
$$(B)\quad 2x-3y+5=0$$

Solution:

$$(A)\qquad y=-x/2-1$$
$$m=-1/2$$
$$(B)\qquad y=(2/3)x+5/3$$
$$m=2/3$$

Since the first line makes the larger angle with the X axis, its slope will be designated as m_2.

$$\text{Tan}\,\beta=\frac{m_2-m_1}{1+m_1m_2}=\frac{-1/2-2/3}{1-2/6}=-7/4$$

Angle β is here an obtuse angle since its tangent is negative. The supplementary acute angle has a tangent of $+7/4$, and the angle itself is readily found from a table of trigonometric functions.

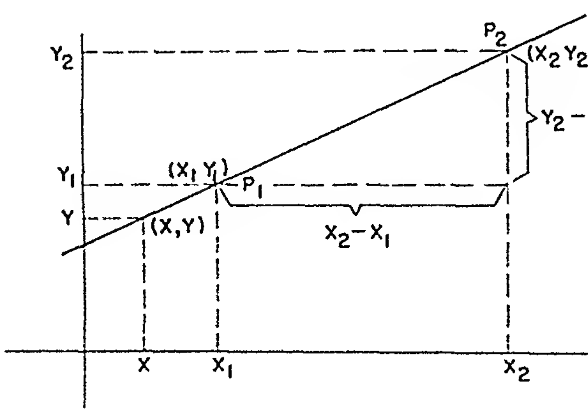
Equation of any Straight Line

The process of establishing an equation from certain known data is now to be described.

If the slope of the Y-intercept of a straight line are known, its equation can be obtained by substituting these values for (m) and (b) in the slope-intercept form of the straight line equation,

$$y=mx+b$$

If two points through which the line passes are known, the equation of the line can be obtained by an application of the fundamental definition of slope.



The points P_1 and P_2 are located by the coordinates (x_1y_1) and (x_2y_2), respectively. The change in the value of the ordinates between these two points is y_2-y_1 and the change in the value of the abscissa is x_2-x_1. The ratio of these two changes is the fundamental definition of the slope of a straight line. Therefore, the slope of the line through these two points is:

$$\frac{y_2 - y_1}{x_2 - x_1}$$

Another point having coordinates (xy) is selected somewhere on the line. The slope at this point is the same at P_1 or P_2, since a straight line has the same slope throughout its length Therefore, the ratio

$$\frac{y_1 - y}{x_1 - x}, \text{ must be equal to the ratio}$$

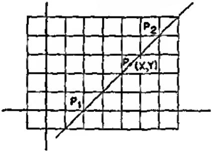
$$\frac{y_2 - y_1}{x_2 - x_1}, \text{ or,}$$

(265). $$\frac{y_1 - y}{x_1 - x} = \frac{y_2 - y_1}{x_2 - x_1}$$

This is known as the *two point form* of the equation of a straight line. It may be applied as follows

Example

Given, P_1 (2,0) and P_2 (6,4) Assume some point P, (x,y) which lies on a straight line, and then substitute in the above equation.



$$\frac{0 - y}{2 - x} = \frac{4 - 0}{6 - 2}$$

$$8 - 4x = -4y, \quad \text{or } x - y = 2$$

If one point through which the line passes and the slope are known, the point slope form of the equation of a straight line may be written. Let (x_1y_1) be the given point and (m) the slope of the line. Another point having coordinates (x,y) is selected somewhere on the line. The slope of the line is then:

$$\frac{y - y_1}{x - x_1}$$

But (m) is also the slope, therefore,

$$\frac{y - y_1}{x - x_1} = m$$

Example

Given, P_1 (4,2) and $m = 2/3$

$$\frac{y-2}{x-4} = 2/3$$

$$2x - 8 = 3y - 6$$
$$2x - 3y - 2 = 0$$

Simultaneous Equations of Straight Lines

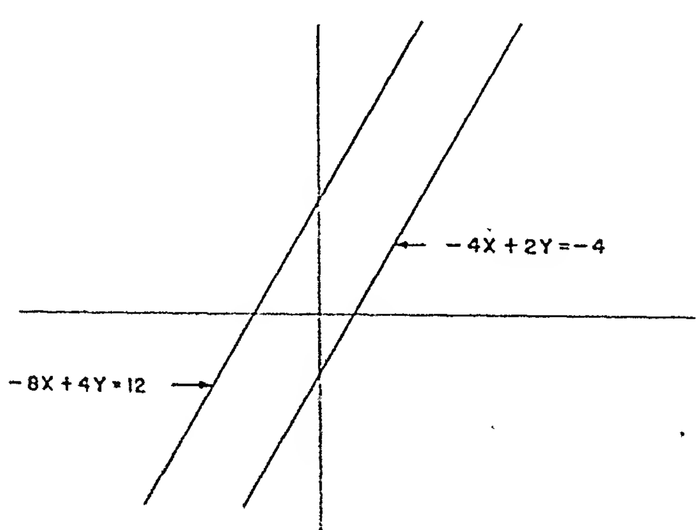
To solve a system of two or more simultaneous equations, it is necessary to obtain a set of values for the variables which will satisfy each of the equations. If the given equations represent straight lines, only one pair of such values can be found, and the equations are termed *independent.* Such equations can be solved graphically or by one or more of the several algebraic methods. However, not every system of equations has a common solution, and it is impossible to find any values for the variables which will satisfy each of the equations. Such equations are called *inconsistent.* In another instance a set of equations may be such that *any* values which will satisfy one of the equations will also satisfy the others. Equations of this nature are called *equivalent* or same equations, because each can be derived from the other.

Whether a system of equations are independent, inconsistent, or equivalent can be determined by graphical means. An alternate procedure is to change the equations to the slope intercept form and observe the nature of their slopes and intercepts.

To fulfill the requirements of *independent* equations, the graph of two straight lines must intersect at just one point. The coordinates of this point are the values of the variables which will satisfy the two given equations. It is apparent, then, that for a solution the lines must intersect, and therefore must not be parallel. In other words, they must be of unequal slope.

The graph of two equations which have the same slope are as shown.

$$\begin{aligned} -4x + 2y &= -4 \\ -8x + 4y &= 12 \end{aligned}$$



Since parallel lines have no points in common, there can be no pair of values satisfying both equations Such equations are inconsistent and cannot be solved Equations of this type are characterized by *equal* slope terms and *unequal* *Y-intercepts.*

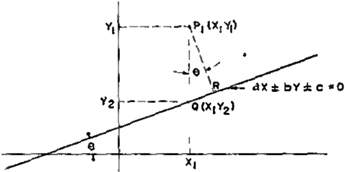
$$\begin{array}{lcl} -4x+2y=-4 & \text{or} & y=2x-2 \\ \underline{-8x+4y=\ \ 12} & \text{or} & \underline{y=2x+3} \end{array}$$

That the *Y-intercepts* must not be identical is apparent, for if parallel lines have the same intercepts the lines are coincident, and all points that lie in one of these lines must also lie in the other. Such equations are *equivalent* and cannot be solved for definite values of the unknowns. Two equations of this type are *identical* when each is reduced to its simplest form, or to the slope intercept form

$$\begin{array}{lcl} x+2y=2 & \text{or} & y=-x/2+1 \\ \underline{3x+6y=6 \text{ or } x+2y=2 \text{ or},} & & \underline{y=-x/2+1} \end{array}$$

Distance From a Point to a Straight Line

The minimum distance from a point to a straight line is the length of the line drawn perpendicular to the line and through the given point



Through point P_1 (x_1y_1) draw a perpendicular to the given straight line $ax \pm by \pm c = 0$, intersecting the line at R. An ordinate line to P_1 intersects the straight line at Q. The ordinate of point Q is y_2 and its abscissa is $x_2 = x_1$ since P_1 and Q are both on the same ordinate line Point Q is on line $ax + by + c = 0$, and its coordinates can therefore be substituted in its equation.

$$ax_1 + by + c = 0$$

or
$$y_2 = -\frac{ax_1 + c}{b}$$

Then $$P_1Q = y_1 - y_2 = y_1 - \left[\frac{ax_1 + c}{b}\right]$$

$$P_1Q = \frac{by_1}{b} + \frac{ax_1 + c}{b} = \frac{ax_1 + by_1 + c}{b}$$

It is evident that the distance P_1Q will be expressed as a positive quantity when P_1 (x_1y_1) lies above the given line, and negative when below. From trigonometry,

the length of the perpendicular P_1R is equal to $P_1Q \cos\theta$. The tangent, and then the cosine, of θ is obtained from the equation of the straight line: By transposing it to the slope intercept form,

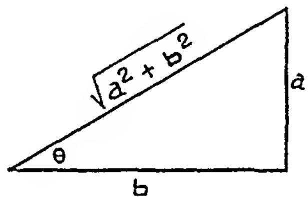
$$y = \frac{a}{b}x - \frac{c}{b}$$

$$\operatorname{Tan}\theta = m = -\frac{a}{b}$$

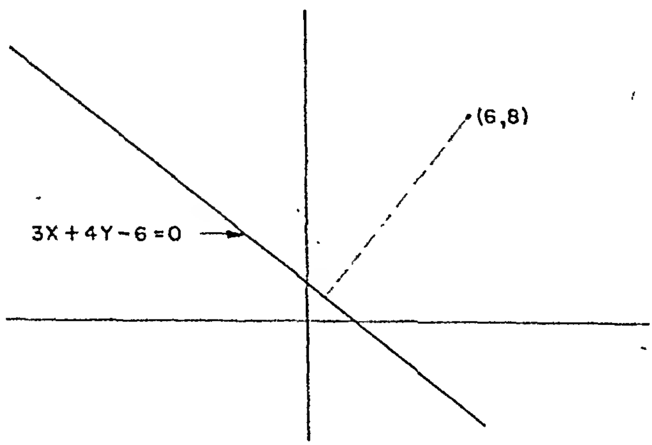
$$\text{Then } \operatorname{Cos}\theta = \frac{b}{\sqrt{a^2+b^2}}$$

$$\text{From which,}\quad P_1R = P_1Q\cos\theta = \left[\frac{ax_1+by_1+c}{b}\right]\left[\frac{b}{\sqrt{a^2+b^2}}\right]$$

(266). $$P_1R = \frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}$$



Example:



$$P_1R = \frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}$$

$a = 3$, $b = 4$, $c = -6$

$x = 6$, $y = 8$

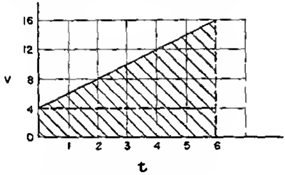
$$P_1R = \frac{(3)\ (6) + (4)\ (8) - 6}{\sqrt{9+16}} = \frac{44}{5} = 8.8$$

The perpendigular distance between any two parallel lines can be found by the same method as for a point and a line. The coordinates of some point P contained in either of the lines is first determined. This is accomplished by assuming some value as abscissa, or ordinate, and then finding the corresponding value of the other coordinate. These values are then substituted in the point-line formula.

Area Beneath a Straight Line Segment

The area beneath a straight line between two points or limits and the X-X axis can be found by a simple computation since the bounded area will be either a rectangle, a triangle, or a combination of the two which is a trapezoid. A problem in uniformly accelerated rectilinear motion demonstrates this point.

> A body in motion is being accelerated at a constant rate. In 6 seconds time its velocity changes from 4 ft/sec. to 16 ft/sec.



$$\text{Acceleration} = a = \frac{16-4}{6} = 2 \text{ ft/sec.}^2$$

$$\begin{aligned} \text{Distance traversed} &= S = \tfrac{1}{2}(16+4)\ (6) = 60 \text{ ft} \\ &= \text{Area beneath straight line.} \\ &= (4)\ (6) + 1/2(16-4)\ (6) \\ &= 24 + 36 = 60 \text{ ft.} \end{aligned}$$

If the value of the average ordinate (average velocity) is desired, it is only necessary to average the ordinate at the first and last point. The average ordinate is also the ordinate at the middle of the graph under consideration

The velocity at any point is represented by the ordinate of the straight-line at the time in question. To determine its value at any time (t) an equation is written by substituting the values of slope and Y intercept in the slope-intercept form of the straight-line.

$$V = 4 + \left[\frac{16-4}{6}\right]t = 4 + 2t \quad \text{or} \quad 2t + 4$$

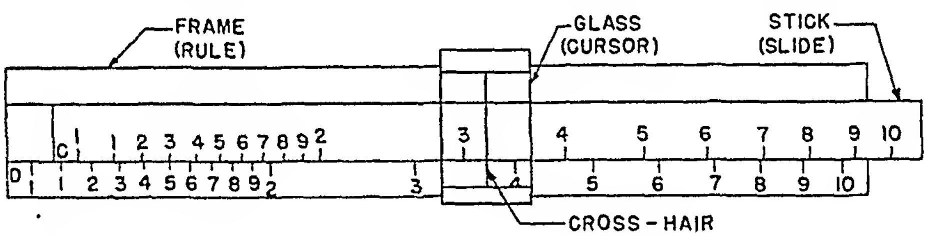
Section VI

APPENDIX — SLIDE RULE

Introduction

The slide rule is an instrument which saves time and labor in performing mathematical computations. The accuracy of the results obtained depend upon the size of the rule employed and the magnitude of the numbers involved. For most practical purposes the inaccuracies involved are not large enough to be of any consequence. For example, the ten-inch rule which is customarily used gives results correct to within one part in one thousand, or one-tenth of one per cent. Greater accuracy can be obtained by using larger rules whenever such precision is required.

A variety of slide rules have been developed which differ in size, arrangement, and also in the method of graduating the scales. However, all rules are used in a similar manner, and the description and method of operation described below for a normal ten-inch rule can be considered to apply, in a general way, to slide rules of all types.



The ordinary slide rule consists of three parts which for convenience will be called the *frame,* the *stick,* and the *glass.* These are more properly named the rule, the slide, and the cursor, respectively, although this terminology does not as readily suggest the part referred to as the names first given. Both the frame and the stick are graduated with various scales of which the *C* and *D* scales are the most used. These two scales are identical and lie adjacent to each other when the stick is installed in the frame with the front of the stick facing forward. The letters *C* and *D* do not always accompany their respective scales, but if employed they may be found at the left-hand edge of the rule and on a line with the scale which they identify.

The entire length of the *C* and *D* scales are graduated into *major* divisions which are numbered from 1 to 10. The extreme right-hand figure may be 1 instead of 10 on some rules. The left-hand digit 1, and the right-hand figure 1 or 10 of the scales are called the left index and the right index, respectively, of the scales on which they appear.

An inspection of the rule reveals that the distance between each two consecutively numbered major divisions is unequal, being largest for the 1 — 2 division, and decreasing progressively for the divisions to the right. Each of these numbered divisions are subdivided into ten *minor* divisions which are not numbered except for those graduations within the first major division. The length of each minor division, between

any two numbers of the major dvisions, varies in the same way as the length of the major divisions vary along the scale.

The figures used in numbering the minor graduations within the first major division are slightly smaller than those used for the major divisions They are also conspicuous by being abbreviated as only the second digit of the appropriate figure is represented. Thus, 1 2 appears as 2, 1 8 as 8, etc. To avoid confusion in reading these numbers, it is recommended that the omitted digit 1 be temporarily marked on the scales until experience justifies its omission The abbreviation of numbers in some parts, and the lack of any numbering system whatsoever for the remaining minor graduations of the rule, is made necessary because of the limited space available For the same reason it is necessary to vary the number, and consequently the value of the smaller graduations which appear between the minor graduations of the rule. For example, on the ordinary ten-inch slide rule, the minor divisions within the first major division (1 — 2) are each subdivided into ten parts Between the major divisions (2 — 3) and (3 — 4) these spaces are divided into five parts The minor divisions throughout the remainder of the scale are divided into two parts only The value to be assigned to any one of these smallest divisions is as defined below.

The position of any number on the *C* and *D* scales of the slide rule is governed by the sequence of digits making up that number, and the position of the decimal point is of no consequence until after the work of the rule has been completed. Thus, the numbers 2 5, 25 and 0 25 all occupy the same position Similarly, the value of the 1 mark on the left-hand of the scale may represent 1, 10, 100, 0 01, etc If, for example, 10 is the value assumed for the left index, then the values for the major graduation numbers throughout the scale becomes 20, 30, 40, etc The numbered minor divisions between 10 and 20 will be read as 11, 12, 13, etc, since in this case the value of each of these divisions is equal to 1 Each of these numbered divisions (11, 12, 13, etc.) are subdivided into ten spaces, consequently the value of each subdivision is 0 1. Assuming that the left index is still considered to be 10, the smallest graduations between 20 and 30 and between 30 and 40 will each represent 0.2 since there are five of these graduations to each minor division which have a value of 1. Between 40 and the right end of the scale the smallest graduations have a value of 0.5 each.

After a small amount of experience is gained in the use of the slide rule, the variation in the value of the smallest divisions appearing on the scales will cause but little difficulty. Whenever doubt exists as to the value to be assigned to any graduation of the slide rule, it is advisable to count between two known points on the scale, one value being less than the unknown value and the other greater In this way the true value of each division is quickly obtained, and from this, the true value of the position in question is determined

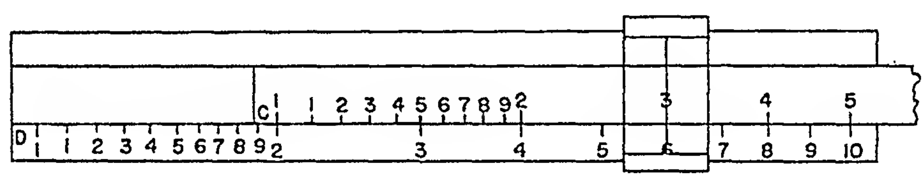
Before undertaking the explanation of how the slide rule may be employed in performing divisions, multiplications, or obtaining square roots of numbers, it is important to have fixed firmly in mind that the results obtained from any of these three operations will always be found on the *D* scale No time and effort need be wasted in searching for answers if this fact is remembered The operation of the slide rule is most easily learned by employing simple examples for which the answers are known. More complex problems can afterwards be tried for practice to obtain accuracy and speed.

FUNDAMENTAL OPERATIONS WITH A SLIDE RULE

Division

The operation of division is the simplest of all computations which can be performed on the slide rule. For the purpose of explanation, the dividend will be referred to as the numerator, and the divisor will be called the denominator. This is a logical procedure for whenever any division is indicated by writing the terms in fractional form, the dividend and the divisor are placed in these respective positions.

In performing divisions, the identical *C* and *D* scales are used. The numerator of the fraction is first set with the glass on the *D* scale. Then the denominator on the *C* scale is set directly above the numerator on the *D* scale, and the quotient is found on the *D* scale at a point directly beneath either the left or the right index of the *C* scale. Thus, to divide 6 by 3 the number 6 is first set on the *D* scale. This is most easily accomplished by moving the glass along the rule until the hair line of the glass coincides with the 6. Then the stick is moved to either the left or right until the 3 of the *C* scale is directly above the 6 on the *D* scale. In this position the two numbers are both in line with the hair line of the glass. The quotient, 2, is then read on the *D* scale at a point directly beneath the left index of the *C* scale.



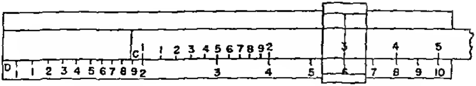
It is helpful to think of the procedure described above as establishing the fraction on the *C* and *D* scales so that the position of the numerator and denominator are interchanged with their normal positions when the desired division is written in fractional form. That is, the numerator is on the *D* scale with the denominator directly above it on the *C* scale.

The position of the decimal point in the result obtained can usually be placed by inspection. Where this is not possible, a rough arithmetical computation will serve to properly locate its position. (Also see POSITION OF THE DECIMAL POINT DETERMINED BY CHARACTERISTICS. Page 213.)

Multiplication

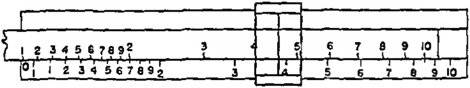
The operation of multiplication is a reverse process to that of division, and consequently the same two scales (*C* and *D*) are employed. For the purpose of explanation it will be assumed that only two factors are being multiplied, for regardless of the actual number, the product of any number of factors is always obtained by multiplying together only two numbers at a time.

To obtain the product of two numbers, set the index, 1, of the *C* scale opposite one of the factors on the *D* scale. Then move the glass along the rule until the second factor is located on the *C* scale. The product is then found directly below on the *D* scale.



Thus, to multiply 2 by 3, set the left index of the *C* scale directly above the 2 on the *D* scale. Then move the glass to the right along the rule until the 3 is located on the *C* scale. The product, which is 6, is then found directly below on the *D* scale.

If the product of the first two digits being multiplied together exceeds 10, it will be found that the second factor, which is to be located on the *C* scale, will project beyond the frame to the right. In this case the stick must be reversed, that is, extended in the opposite direction, so that the right index instead of the left is used. With the right index placed directly above one of the factors on the *D* scale, the glass is moved to the left until the second factor is located on the *C* scale. The product is then found directly below on the *D* scale.



Thus, to multiply 45 by 85, set the *right* index of the *C* scale directly above 85 on the *D* scale. Then move the glass to the left until the 45 is located on the *C* scale. The product, which is 3825, is then found directly below on the *D* scale. Whether to use the left or the right index of the *C* scale when performing multiplications is of no great importance, as the fact is soon discovered when the opposite index must be employed. However, to effect a saving in time and labor, the following rule will be found useful in determining which index to use.

> If the product of the first figures of each of the given numbers is less than 10, use the left index, if this product is greater than 10, use the right index.

An exception to this rule will be found in such cases as 3.12 times 3.31. According to the rule the left index should be used. It will be found, however, that it is necessary to use the right index. This is due to the fact that although the product of the first digits of the two numbers is less than 10, the product of the complete numbers is greater than 10.

As seen from the examples given, the rule is not without exception. However, in most cases the rule applies, and its consistent use will save an appreciable amount of time and effort if numerous computations are to be made.

Combined Multiplication and Division

For continued operations involving both multiplication and division such as

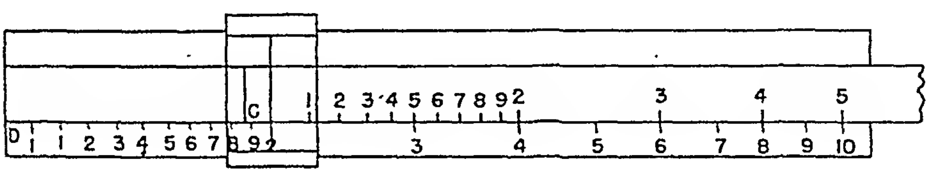
$$N = \frac{(a)\ (b)\ (c)}{(x)\ (y)\ (z)}$$

proceed by dividing the first factor in the numerator by the first divisor in the denominator. Next multiply the quotient obtained by the second factor in the numerator and then divide by the second factor in the denominator, etc., thus progressing through the problem according to a saw-tooth plan. Solving the problem in this manner often eliminates several settings and readings of the slide rule, thereby lessening the labor involved and diminishing the possibilities of errors.

In the explanation above, it was stated that the first factor of the numerator was to be divided by the first divisor of the denominator, etc. However, the order in which these terms are used is unimportant provided that the numbers are chosen alternately from the numerator and the denominator. Experience with such problems will reveal that the order in which the values are given is not always the most practical order in which the problem can be solved. Another fact of importance is that if there are more factors in the numerator than there are divisors in the denominator, or vice versa, it is possible to use 1 as a factor or divisor as many times as may be necessary.

Proportions

If the left index of the *C* scale is placed directly above the number 2 of the *D* scale, the value of the ratio (fraction) established will be 1/2. Other ratios having the same value will be found to exist at all adjacent points along the *C* and *D* scales.



Other arbitrary positions of the two scales will verify the fact that for any position of the stick, the numbers of the *C* scale bear a constant ratio to those on the *D* scale. Thus, to solve a proportion such as

$$\frac{x}{62.4} = \frac{231}{1728} \text{ which is the same as } \frac{231}{1728} = \frac{x}{62.4}$$

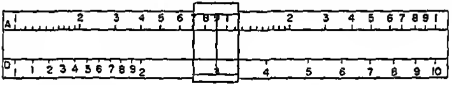
it is only necessary to place 231 on the *C* scale directly above 1728 of the *D* scale. Then look along the *D* scale for 62.4. The numerical value of (x) will be found directly above the 62.4. In this case the value of (x) is 8.34.

Other problems in proportion are similarly solved. This method of procedure is of advantage in that the operation of cross-multiplying is eliminated, thus saving time, labor and possibility of error.

Square Roots and Squares of Numbers

The square root of a number is a quantity which when multiplied by itself, will produce the original number. Thus, $3 = \sqrt{9}$, since $(3)(3) = 9$. The operation of finding the square root of a number involves the use of the *A* and the *D* scales Since neither of these scales are found on the stick, it may be removed from the rule when obtaining square root, thus eliminating a possible source of confusion The *A* scale can be quickly identified on the rule since it consists of two identically graduated scales, placed consecutively, at the top of the frame. The total length of these two scales is the same as the full length of one *C* or *D* scale and each is graduated in a similar manner, although the subdivisions are fewer in number and consequently have a larger value. In many cases, the *A* scale is on the front of the rule so that the four visible scales are the *A*, *B*, *C*, and *D*, reading from the top to the bottom. However, on some rules the *A* scale is on the side opposite to that of the *C* and *D* scales In these cases, the *D* scale of the rule is repeated on the back side of the frame so that in all cases there is a *D* scale directly below the *A* scale

Each half of the *A* scale is graduated in a manner similar to the *D* scale, but since the former is only one-half as long, the graduations are fewer and consequently have a larger value The position of any number can be established as easily on the *A* scale as on the *C* or *D* scales if the method of counting as described on page 205 is employed



The square root of any number located on the *A* scale is found at a point on the *D* scale directly below that number For locating this point on the *D* scale the cross-hair of the glass is invariably employed. Since the *A* scale consists of two equal and identical sections, it becomes necessary to determine on which one of the two scales the given number should be located. Of the several rules available for governing this decision, the most infallible seems to be to count the number of digits occurring to the left of the decimal point in the given number If this *number* of digits is odd, such as 1, 3, 5, etc., the left hand section is used, if the number of digits is even, such as 2, 4, 6, etc, the right-hand section is used

The position of the decimal point in the answer read from the *D* scale is usually apparent and no question is involved in its determination However, it may be definitely located in any case by placing the decimal point in the result so that there is one whole number figure in the square root for each group of two digits occurring to the left of the decimal point in the given number. The groups of two figures referred to are called periods The last period so marked off may consist of either one or two digits For additional description see paragraphs entitled **Square Root of Numbers** included in SECTION I.

The above rules for determining which section of the *A* scale is to be employed in finding the square root of a number are obviously not applicable to numbers which are decimal fractions, since, in this case there are no digits to the left of the decimal

point. To obtain the square root of any decimal fraction, first move the decimal point in the given number an *even* number of places to the right of its original position so that the value of the fraction becomes some number between 1 and 100. The resulting number may be a whole number, or a whole number accompanied by a decimal as shown below. Now find the square root of the number thus formed, and in the answer move the decimal point to the left one-half as many places as it was moved to the right in the first operation. The resulting number is the square root of the given decimal fraction.

$$\sqrt{0.0625}$$
$$\sqrt{006.25} = 2.5$$
$$0.0625 = .25 \text{ or } 0.25 \text{ Ans.}$$

The operation of squaring a number can be accomplished by a reverse process to that described above. Thus, the square of any number is found at a point on the *A* scale directly above that number located on the *D* scale. For locating this point on the *A* scale, the cross-hair of the glass is invariably employed.

The method of squaring a number by the use of the *A* and *D* scales is in addition to the method of multiplication on the *C* and *D* scales in which case the given number is simply multiplied by itself.

Square Root of the Sum of the Difference of Two Squares

The need for calculating the square root of the sum of two squares or the square root of the difference of two squares is frequently encountered. If these operations are performed with the use of a slide rule, it is advisable to factor out the square root of one of the terms beneath the radical sign before the square root of the sum or the difference of the squares of the terms is obtained. This procedure makes it possible to deal with much smaller numbers with a resulting saving in time and effort. This is evident from an examination of the examples which follow.

$$\begin{aligned}\sqrt{90^2+360^2} &= 90\sqrt{\frac{90^2}{90^2}+\frac{360^2}{90^2}} = 90\sqrt{1^2+4^2}\\ &= 90\sqrt{1+16} = 90\sqrt{17} = 90\times 4.12\\ &= 371\end{aligned}$$

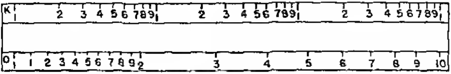
$$\begin{aligned}\sqrt{120^2-40^2} &= 40\sqrt{\frac{120^2}{40^2}-\frac{40^2}{40^2}} = 40\sqrt{3^2-1^2}\\ &= 40\sqrt{9-1} = 40\sqrt{8} = 40\times 2.83\\ &= 113\end{aligned}$$

The operations as shown above in detail are not necessarily written down when a slide rule is being used.

Cube Roots and Cubes of Numbers

On some slide rules a scale may be found which is made up of three identically graduated scales placed consecutively and at the top of the frame. The total length of these three scales is the same as the full length of one *C* or *D* scale and each are graduated in a similar manner, although the subdivisions are fewer in number and consequently

have a larger value. The three consecutively placed scales are referred to collectively as the K scale. It is usually distinctly marked by a letter K found at the left-hand edge of the rule and on a line with the scale it identifies. Each of the three divisions making up the K scale is referred to as K_1, K_2, and K_3 respectively, reading from left to right



The cube root of any number located on the K scale is found at a point on the D scale directly below that number For locating this point on the D scale, the cross-hair of the glass is invariably employed Since the K scale consists of three equal and identical sections, it becomes necessary to determine on which of the three the given number should be located Of the several rules available for governing this decision, the most infallible seems to be

Use K_1 for a number of 1 digit to the left of the decimal point.
Use K_2 for a number of 2 digits to the left of the decimal point.
Use K_3 for a number of 3 digits to the left of the decimal point

The decimal point in the cube root of all numbers from 1 to 100 is placed so that there is one whole number in the root Thus, the cube root of 5, 15, and 512 are 1.71 . , 2 46 . . , and 8 00 respectively

The above rules for determining which section of the K scale is to be employed in finding the cube root of a number are obviously not applicable to numbers which lie outside the range of numbers of 1 to 1000 To find the cube root of any number less than 1 or greater than 1000, it is necessary to first move the decimal point 3, 6, 9, or any multiple of 3 places to either the left or the right from its original position so that the value of the number to the left of the decimal point becomes some number between 1 and 1000 Now find the cube root of the number thus formed, and then move the decimal point one-third as many places as it was moved in the first place, but in the opposite direction The resulting number is the cube root of the given number.

$$\sqrt[3]{0\ 027}$$
$$\sqrt[3]{0027.} = 3\ 00$$
$$\sqrt[3]{0\ 027} = 3 \text{ or } 0\ 3 \text{ Ans}$$

The operation of cubing a number can be accomplished by a reverse process to that described above Thus, the cube of any number is found at a point on the K scale directly above that number located on the D scale For locating this point on the K scale the cross-hair of the glass is invariably employed.

The method of cubing a number by the use of the K and D scales is in addition to the method of multiplication on the C and D scales in which the given number is simply multiplied by itself two times.

Trigonometric Functions

The slide rule provides a convenient means by which the natural trigonometric ratios sine and tangent for any acute angle can be read directly from the appropriate scale. The values of the cosine function for the various angles are not given on the rule, but they may be indirectly determined in any case by finding the sine of the angle which is the complement of the given angle. One angle is said to be the complement of another angle if their sum is 90°. Thus, the slide rule may be said to provide all three of the ratios—sine, cosine, and tangent.

An inspection of the back of the stick shows that it is graduated with three scales, one being a scale of sines, indicated by the letter *S*; another a scale of tangents, marked *T*; and the third scale will be either a *B* or an *L* scale. The use of the *L* scale is described in later paragraphs entitled Logarithms.

The *S* and the *T* scales are both graduated in degrees and fractions of a degree, called minutes. The divisions representing degrees are clearly marked, but the subdivisions representing minutes are not. Since each subdivision may represent a number of minutes, it becomes necessary to determine the valu of these graduations when reading the scales. It should be remembered that 1 degree is equal to 60 minutes. A position on the scale intermediate between two numbered graduations as 5° and 6° is read as 5° 30′. If there are, for example, six divisions between two adjacent degree graduations, then each subdivision represent 10′.

On the *S* scale, the graduations extend from 0° to 90°, but on the *T* scale, the range of angles is from approximately 6° to 45°. This limitation of the angles on the *T* scale is made necessary as the tangent function varies over such a wide range, being 1 for 45° and becoming approximately 4000 as 90° is approached. Consequently, it would be impossible to represent the entire range with any degree of accuracy in the limited space available.

The sine or the tangent of an angle is most easily found on any slide rule by installing the stick in the rule so that the *A* or *B*, *S*, *T*, and *C* or *D* scales are all on the same side. This requirement may make it necessary to have the back side of the stick facing forward in some slide rules. The *A* and *B* scales are identical, therefore they may be used interchangeably. This is also true for the *C* and *D* scales.

Sine of an Angle

With the end graduations of the *S* scale exactly in line with the end graduations of the *A* or *B* scales, the glass may be moved to any position along the *S* scale, and the sine of the angle thus marked will be indicated by the cross-hair of the glass at a point directly above on either the *A* or *B* scale. The values of the sines found on either the *A* or *B* scale vary from 0.01 to 0.1 on the left half of the scale, and from 0.1 to 1.0, on the right half. Thus, it is important to realize that for angles having sines read on the left half of the scale (0° to 5° 45′) the decimal point must be followed by one zero when the position of the decimal point of the function is being determined. For angles having sines read on the right half of the scale, the sequence of digits representing the function immediately follows the decimal point.

Another method of finding the sine of an angle can be employed when using the

type of slide rule which does not require the stick to be reversed for the operation as previously described With the S scale facing rearward in its normal position, the stick is drawn out to the right until the desired angle is in line with the cross-hair of the index mark of the part cut out of the back of the frame The rule is then turned around and the sine is found on the B scale at a point directly below the right index of the A scale

Cosine of an Angle

The cosine of an angle is not given directly on the slide rule, but the value of this function for any angle may be determined indirectly by finding the sine of the angle which is the complement of the given angle.

Tangent of an Angle

With the end graduations of the T scale exactly in line with the C or D scales, the glass may be moved to any position along the T scale, and the tangent of the angle thus marked will be indicated by the cross-hair of the glass at a point directly below on either the C or D scale The sequence of digits representing the function immediately follows the decimal point as the lower limit of the T scale excludes all angles of such size as to have their tangents measured in hundredths The tangents of angles below this lower limit are replaced by the corresponding sines without introducing any appreciable error. The tangents of angles larger than 45° can be found by either one of two methods

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\tan\theta = \frac{1}{\tan(90° - \theta)}$$

/ Another method of finding the tangent of an angle can be employed when using the type of slide rule which does not require the stick to be reversed for the operation as previously described. With the T scale facing rearward in its normal position, the stick is drawn out to the right until the angle in question is in line with the cross-hair of the index mark of the part cut out of the back. The rule is then turned around and the tangent is found on the C scale at a point directly above the right index of the D scale.

Logarithms

The complete logarithm of any number consists of two parts called the characteristic and the mantissa. The characteristic is that part of the logarithm to the left of the decimal point, and its magnitude is determined by a mental calculation described in the paragraphs titled RULES FOR CHARACTERISTICS which is included in SECTION IV. The mantissa is the decimal part of the logarithm, or that part to the right of the decimal point The value of the mantissa is the same for any given sequence of digits, regardless of the position of the decimal point in that sequence. Thus, the complete logarithms of 12 34 and 1234 are 1 091315 and 3.091315 respectively. It is obvious that the mantissas of these two logarithms are identical,

although the characteristics are of different value, being dependent upon the position of the decimal point in the given number.

The mantissa of the logarithm of any number is read on the *L* scale after properly locating the number on the *D* scale. The entire length of the *L* scale is graduated into major divisions of equal length and numbered from 1 to 10. Each of these graduations are subdivided into ten equally spaced minor divisions which in turn are each subdivided into ten smaller divisions. The length of the scale may, therefore, be said to be graduated decimally, and no difficulty should be encountered in determining the value of any subdivision, or any position on the scale.

Because the scales of the various slide rules are not similarly arranged, it is necessary to describe several possible methods of obtaining the mantissa for any given number.

If the *D* and the *L* scales are both found on the same side of the rule when the stick is installed with its front face forward, the mantissa is read on the *L* scale at a point directly above the position of the number on the *D* scale. The cross-hair of the glass when moved to the position of the number on the *D* scale accurately locates the mantissa directly above on the *L* scale.

A few slide rules are operated by the same method as described above after installing the stick in the frame with its back side forward. Such rules can be identified, as the numbered divisions of the *L* scale will increase from left to right when the stick is installed in the manner described.

If the *L* scale is one of the three scales found on the back side of the stick, and the numbered divisions decrease from left to right, the mantissa is read from the *L* scale when the left index of the *C* scale is placed directly above the value of the number on the *D* scale. A cross-hair is usually provided in the back and at one end of the frame of the rules which are operated in this manner. The *C* and *D* scales are obviously on the side of the rule opposite to the *L* scale; consequently the rule must be turned over to read the mantissa.

POSITION OF THE DECIMAL POINT DETERMINED BY CHARACTERISTICS

The decimal point in the result of all operations involving multiplication or division performed on the *C* and *D* scales can be definitely located by employing the characteristic of the logarithms of the numbers involved. The characteristic of each of the terms of the problem is first written down somewhere adjacent to the number. Then the slide rule work is performed according to the methods already described for multiplication, division, and combined operations. For each separate operation in the solution it is necessary to observe the position of the left index (1) of the *C* scale. If this index projects beyond the left index of the *D* scale, the characteristic of the term which caused it to extend should be increased by 1. When the index does not project, the characteristic of the term used remains unchanged. The characteristic used in placing the decimal point in the answer is the algebraic difference of the characteristics of the terms constituting the dividend (numerator) and the characteristics of the terms comprising the divisor (denominator).

For continued operations it is essential that the position of the left index of the *C* scale be noted as each new term is established on the rule, and any necessary addition to the characteristic be immediately made. Not only is the characteristic of the term in-

creased which caused the left index to first project, but also the characteristics of the following term or terms are also increased if they allow the index to remain extended to the left as such terms are employed.

$$\frac{\overset{1}{45}}{\underset{1}{15}} = \overset{1-1}{3} = \overset{0}{3} \text{ or } 3.$$

$$\frac{\overset{1}{50}}{\underset{1+1}{75}} = \overset{1-2}{666\,..} = \overset{-1}{666\ldots} = .666\quad .$$

$$\frac{\overset{-1}{.25} \times \overset{1+1}{24}}{\underset{1+1}{50} \times \underset{0+1}{5}} = \overset{(1)-(3)}{24} = \overset{-2}{24} = .024$$

Section VII

ALGEBRA — PROBLEMS

Positive and Negative Numbers

1. Find the sum of the following.

(a)	(b)	(c)	(d)	(e)	
-1	-5	-9	$+13$	-3	(f) $10+(-14)=$
$+2$	-6	$+10$	$+14$	$+8$	(g) $.3+(-.08)=$
-3	$+7$	-11	-15	16	(h) $-1.2+6=$
$+4$	$+8$	$+12$	-16	-3	

2. Find the difference when the lower number is subtracted from the upper number in each column.

(a)	(b)	(c)	(d)	(e)	(f)
5	3	5	-5	5	-8
3	5	-3	3	10	-3

3. Find the product of all the numbers in each column.

(a)	(b)	(c)	(d)
-1	5	-1	5
2	-6	-6	2
3	-7	3	-7
4	8	-4	8

4. Find the quotient of the following.

(a) $\frac{16}{-4}$ (b) $\frac{-12}{-3}$ (c) $\frac{144}{-16}$ (d) $\frac{56}{8}$ (e) $\frac{175}{-5}$ (f) $\frac{144}{-6}$

Common Fractions

5. *Addition*

(a) $\frac{2}{5}+\frac{3}{5}=$ (b) $\frac{2}{3}+\frac{3}{4}=$ (c) $\frac{2}{3}+\frac{3}{4}+\frac{4}{5}=$ (d) $\frac{x}{4}+\frac{y}{6}+\frac{z}{12}=$

6. *Subtraction*

(a) $\frac{3}{5}-\frac{2}{5}=$ (b) $\frac{5}{6}-\frac{1}{3}=$ (c) $\frac{3}{4}-\frac{1}{8}=$ (d) $\frac{x}{5}-\frac{y}{3}=$

7. *Multiplication*

(a) $\frac{1}{2}\times\frac{2}{3}=$ (b) $2\times\frac{5}{4}=$ (c) $\frac{13}{25}\times\frac{25}{13}=$ (d) $\frac{x}{4}\times\frac{2}{x}=$

8. *Division*

(a) $\frac{3}{5}\div\frac{4}{5}=$ (b) $\frac{5}{4}\div 2=$ (c) $\frac{6}{5}\div 2=$ (d) $\frac{x}{2}\div\frac{x}{4}=$

9. Find the reciprocals of the following.

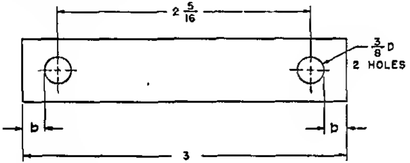
(a) $\frac{1}{2/3} =$ (b) $\frac{1}{1/3} =$ (c) $\frac{1}{1/3 + 1/2} =$ (d) $\frac{1}{a/b} =$

10. Least common denominator.

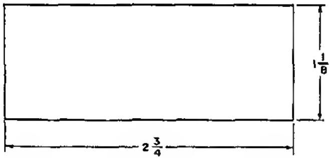
(a) $\frac{5}{24} + \frac{3}{15} + \frac{2}{9} =$ (b) $\frac{2}{3} + \frac{1}{4} - \frac{1}{6} =$ (c) $\frac{x}{a} + \frac{y}{b} =$ (d) $\frac{x}{a} - \frac{y}{b} =$

11. Mixed numbers.
(a) $1\frac{1}{2} + 2\frac{3}{4} =$ (b) $5\frac{1}{8} - 3\frac{3}{4} =$ (c) $2\frac{2}{3} \times 1\frac{5}{8} =$ (d) $1\frac{2}{3} \div \frac{3}{8}$

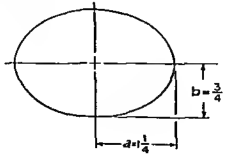
12. Find the dimension (b).



13 Find the area of the rectangle.



14. Find the area of the ellipse with a semi-major axis (a) of $1\frac{1}{4}''$ and a semi-minor axis (b) of $\frac{3}{4}''$.



15. The gross (total) weight of a given airplane is 10,000 lbs., its weight empty is 60% of its gross weight. The crew weighs 170 lbs., fuel and oil weigh 1,000 lbs. and remainder is to be freight. How much freight can be carried by the airplane.

Decimal Fractions

16. Find the *sum* of all the numbers in each column.

(a)	(b)	(c)	(d)
.125	.09375	2.25	0.125
.250	.0625	.032	2.500
.375	.345	3.278	1.075

17. Find the difference when the lower number is subtracted from the upper in each column.

(a)	(b)	(c)	(d)	(e)	(f)
2.50	3.75	1.118	1.50	.25	6.25
1.125	1.0625	.219	.15	.125	3.875

18. Find the product of all the numbers in each column.

(a)	(b)	(c)	(d)
5.625	3.1416	3.25	17.25
.05	.25	1.25	3.02

19. Find the quotient when the upper number is divided by the lower.

(a) $\dfrac{3.1416}{4}$ (b) $\dfrac{2.5}{.0625}$ (c) $\dfrac{22.7766}{3.1416}$ (d) $\dfrac{23.625}{1.05}$

20. Find the decimal equivalents of the following.

(a) $\frac{1}{3} =$ (f) $\frac{1}{8} =$

(b) $\frac{1}{4} =$ (g) $\frac{1}{9} =$

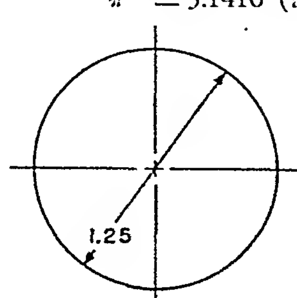
(c) $\frac{1}{5} =$ (h) $\frac{1}{16} =$

(d) $\frac{1}{6} =$ (i) $\frac{1}{32} =$

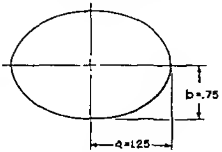
(e) $\frac{1}{7} =$ (j) $\frac{1}{64} =$

21. Find the circumference of a 1.25 inch diameter circular shaft.

$$\text{Circumference} = \pi D = 3.1416 \times 1.25$$
$$\pi = 3.1416 \text{ (approx.)}$$

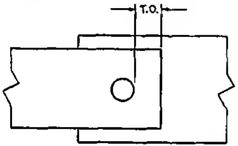


22. Find the area of an ellipse with a semi-major axis (a) of 1.25", and a semi-minor axis (b) of .75"



$$\text{Area} = \pi ab = 3.1416 \times 1.25 \times .75$$

23. The tear-out strength (P_t) of a riveted or bolted joint can be computed by the use of the formula

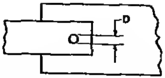


$$P_t = F_s \times 2T.O. \times t$$

Where P_t = tear-out strength of sheet (lbs.)
F_s = allowable shear stress of sheet material (lbs/sq. in.)
$T.O.$ = tear-out distance or the distance from the edge of the rivet or bolt hole to edge of the sheet (ins.)
t = thickness of sheet (ins.)

Find the tear-out strength of a lap joint for which
$F_s = 34{,}000$
$t = 040$
$T.O. = 25$

24. The bearing strength (P_{br}) of a riveted or bolted joint can be computed by the formula

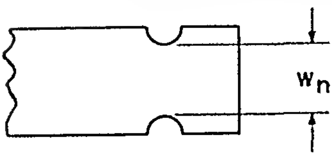


$P_{br} = F_{br} \times D \times t$ Where P_{br} = bearing strength of sheet (lbs)
F_{br} = allowable bearing stress of sheet material (lbs./sq in.)

Find the bearing strength of a joint for which

$$F_{br} = 75{,}000 \ p.s.i.$$
$$D = 3/32$$
$$t = .040$$

25. The tensile strength (P_t) of a riveted or bolted joint can be computed by the use of the formula



$$P_t = F_t \times W_n \times t$$

Where P_t = tensile strength of sheets (lbs.)
F_t = allowable tensile stress of sheet material (lbs./sq. in.)
W_n = net width of sheet (in.)
t = thickness of sheet (ins.)

Find the tensile strength of a joint where

$$F_t = 56{,}000 \ p.s.i.$$
$$t = \tfrac{1}{2}''$$
$$W_n = .10$$

Square Root of Numbers

26. Solve the following.

(a) $\sqrt{17424}$

(b) $\sqrt{11296321}$

(c) $\sqrt[3]{8}$

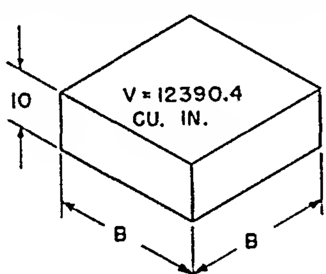
(d) $\sqrt[4]{256}$

(e) $\sqrt{\frac{36}{64}}$

27. Find the radius of a circle which has an area of 28.2744 sq. in.

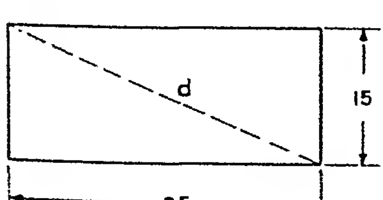
$$\text{Area} = \pi r^2 = 3.1416\, R^2$$

28. Find the length of the side of a box which has a square bottom and a height of 10 inches. The volume of the box is 12390.4 cu. in.



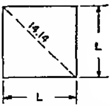
$$\text{Volume} = B \times B \times 10$$

29. The length of the diagonal of a rectangle is given by the Pythagorean Theorem as $d = \sqrt{a^2 + b^2}$ where a and b are the dimensions of the two sides. Find the length of the diagonal when $a = 15$ inches and $b = 25$ inches.



$$d^2 = 15^2 + 25^2$$
$$d = \sqrt{15^2 + 25^2}$$

30. Find the length of the side of a square which has a diagonal of 14.14 inches.

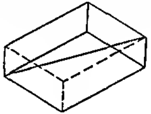


$$14.14^2 = L^2 + L^2 = 2L^2$$

$$L^2 = \frac{14.14^2}{2}$$

$$L = \sqrt{\frac{14.14^2}{2}}$$

31. The length of the longest diagonal of a rectangular solid is given by the formula $d = \sqrt{a^2 + b^2 + c^2}$ where a, b, and c are the dimensions of the three sides of the solid. Find the length of the longest diagonal when $a = 8$ inches, $b = 12$ in., $c = 16$ in.



$$d = \sqrt{a^2 + b^2 + c^2}$$

Exponents

32. Find the product of all factors in each line and the numerical product.

(a) $(x)\,(x) =$ Let $x = 3$
(b) $(x^1)\,(x^2) =$ Let $x = 2$
(c) $(x^4)\,(x^3) =$ Let $x = 2$
(d) $(x^{1/2})\,(x^{1/4}) =$ Let $x = 3$

33. Solve the following.

(a) $\frac{3x^2}{2} + \frac{8x^2}{5} =$

(b) $(2x)^2 =$

(c) $(5y)^3 =$

(d) $\frac{(6y)^2}{-3y} =$

34. Find the product of all factors in the following.

(a) $(x)\,(x^1)\,(x^2)\,(x^3) =$
(b) $(x^2)\,(x^3)\,(x^4)\,(x^5) =$
(c) $(x^{1/2})\,(x^{3/4})\,(x^{1/4})\,(x^{1/4}) =$
(d) $(2x)\,(3x^2)\,(4x^3)\,(5x^4)$
(e) $\frac{(x)}{2}\,\frac{(3x^2)}{4}\,\frac{(5x^3)}{2}\,\frac{(4x^4)}{5}$

35. Simplify the following.

(a) $\frac{x^3}{x^2} =$

(b) $\frac{x}{x^{-2}} =$

(c) $\sqrt{16x^4y^2}$

(d) $\sqrt[3]{\frac{8x^4y^5}{x^{-5}y^2}} =$

36. Solve the following

(a) $\sqrt{a^2} =$
(b) $\sqrt[2]{9y^2} =$
(c) $(4a^2)b^2 =$

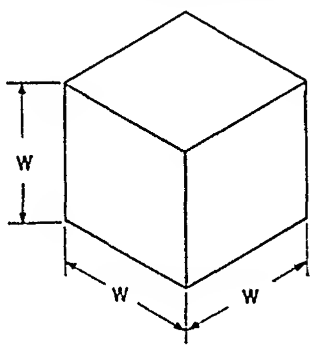
37. (a) $(x+y)(x+y) =$ (d) $(x)(x)^6 =$
(b) $\sqrt{8x^2 + (3x)(-2x)} =$ (e) $(2^3)(3)^2 =$
(c) $(a^2)(a^3) =$ (f) $3^2 - 4^3 =$

38. (a) $x^3 \div x^2$ (d) $a^{2.5} \div a^{.5} =$
(b) $4^3 \div 2^5$ (e) $(x^{1.5})(x^2) =$
(c) $\dfrac{\sqrt{x^3}}{\sqrt{x^5}} =$

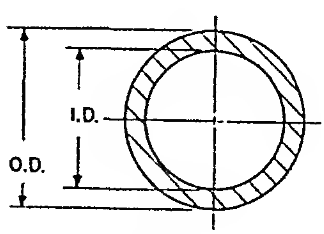
39. (a) $\sqrt{8^4} =$ (c) $\sqrt[4]{2^8} =$
(b) $\sqrt[3]{7^4} =$ (d) $\sqrt[5]{a^n} =$

40. (a) $(x)^2(xy^4) =$ (d) $-4^2 \div 2^4 =$ (g) $\dfrac{a^2}{a^{-3}}$
(b) $(3a)^2(3^2a) =$ (e) $a^2 \div b^2 =$
(c) $a^3b^3 \div a^5b^5 =$ (f) $\dfrac{x^{-3}}{y^{-2}} =$

41. (a) Find the surface area of a cube which measures (w) on any side.
(b) If $w = 6$, find the surface area of the cube.



42. Find the cross sectional area of a one inch circle having a wall thickness of .0625 in.



$$\text{Area of Circle} = \pi R^2 = \pi \left(\frac{D}{2}\right)^2 = .7854\, D^2$$

$$\text{Since } R = \frac{D}{2},\ R^2 = \left(\frac{D}{2}\right)^2 = \frac{D^2}{4}$$

$$\text{Area of tube} = \frac{\pi\, \overline{O.D.}^2}{4} - \frac{\overline{I.D.}^2}{4} = .7854\, \overline{O.D.}^2 - .7854\, \overline{I.D.}^2$$

Solution of Equations

Solve for the unknown (x) in each of the following equations. Check the results obtained to see that the given equation is satisfied.

43 $x-2=0$

44 $x+3=18$

45. $x-5=3$

46. $2x+7=14+x$

47 $5x=10+4x$

48. $x-6=4x$

49 $x+2=2x-6$

50 $x^2+5x-x^2-4x-4=4$

51 $x^2-3+2x=x^2-x$

52 $x^2+2x-5=x^2+x$

Multiplication—Division

53 $2x=10$

54 $5=\frac{x}{2}$

55 $3=\frac{12}{x}$

56 $\frac{2x}{3}=6$

57 $4x+3=6x-8$

58 $3x+3=18$

59 $6x-4=4x+6$

60 $5x=35-2x$

61 $5x=0$

62 $4x^2=44x$

63 The aspect ratio of a monoplane airplane wing is expressed by the formula:

$$\overline{AR}=\frac{b^2}{S}$$

Where $\overline{AR}$ = The aspect ratio (a non-dimensional number)
b = Wing span in feet
s = Wing area in square feet

For an aspect ratio of 10, find the required span if the wing area equals 300 square feet

Method of Solution—Single Equation

Simplification—solve for x in each of the following

64. $33x=(30)(60)-(5)(48)-(10)(24)$

65 $10x-(x-9)=35-4(2x-1)$

65. $x-(3-2x)=2x-4$

67 (a) $3(x+1)+4=5(x-3)$
(b) $6(x-b)=12+5(x+b)$

68. (a) $12x=\frac{x-6}{3}$
(b) $\frac{12}{x-3}=\frac{4}{x+1}$
(c) $(2x-1)-2\,[4x-(x+2)+10]=8x+1$

69. (a) $x+(3-a)=3x+(1+2a)$
(b) $a-(x+2)-(a+3)=x+4(a+1)-(x+5)$

70. (a) $\frac{2}{3}x+\frac{3}{4}x=4$ (b) $3+\frac{2x+4}{4x-2}+\frac{2}{x+2}=6$

71. The equation for finding the bending stress f_b in a beam is

$$f_b = \frac{MC}{I}$$

In a beam of rectangular cross section $I = \frac{bh^3}{12}$ and $c = \frac{h}{2}$

Write the complete equation for computing the stress in this beam in terms of b and h.

72. The aspect ratio (AR) of the wing of airplane $= \frac{b^2}{s}$ where $s = bc$.

Write the equation for AR.

73. The formula for converting the readings of Fehrenheit temperature to Centigrade is:

$$C^\circ = \frac{5}{9}(F^\circ - 32^\circ)$$

Find the temperature in Centigrade degrees which is equivalent to 86° F.

74. The formula for converting the readings of Centigrade temperatures to Fahrenheit temperatures is:

$$F^\circ = \frac{9}{5}C^\circ + 32^\circ$$

Find the temperature in Fahrenheit degrees which is equivalent to 40° C.

75. The net cross sectional area of a circular tube can be computed by the use of the following formula:

$$A = \pi t\,(O.D. - t)$$

Where A = cross sectional area

$\pi = 3.1416$ (approx.)

$O.D.$ = outside diameter of tube

t = wall thickness of tube

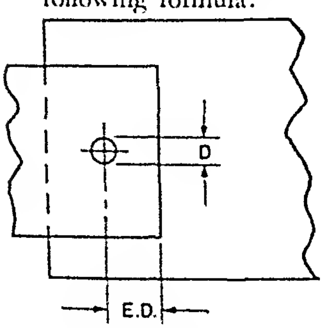
Note: Median Diameter $= O.D. - t$

Circumference of Median Line $= \pi\,(O.D. - t)$

Area $= \pi\,(O.D. - t)t = \pi t(O.D. - t)$

Find the net cross sectional area of a .75 $O.D. \times$ .065 tube.

76. The tear-out strength of a riveted joint can be computed by the use of the following formula:



$$P_s = 2F_s t\,\left(E.D. - \frac{D}{2}\right)$$

Where P_s = Tear-out strength of sheet

F_s = Allowable shear stress of sheet

t = Thickness of sheet

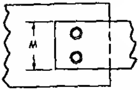
$E.D.$ = Distance from center of rivet to edge of sheet.

D = Diameter of rivet

Find the tear-out strength of a riveted lap joint for which

$F_s = 34{,}000 \quad t = .040 \quad E.D. = 25 \quad D = \frac{3}{16}$

77 The tensile strength of a riveted joint can be computed by the use of the following equation:



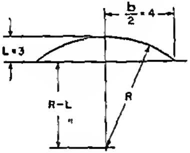
$$P_t = F_t t(W - nD)$$

Where P_t = net tensile strength of sheet
F_t = allowable tensile stress in sheet
t = thickness of sheet
W = width of sheet
n = number of rivets
D = diameter of rivets

Find the net tensile strength of a riveted lap joint for which:

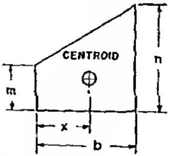
$F_t = 56{,}000 \quad t = 040 \quad W = 1.5 \quad n = 2 \quad D = \frac{3}{16}$

78 Find R if $L = 3$ and $\frac{b}{2} = 4$ Hint. $(R - L)^2 + \left(\frac{b}{2}\right)^2 = R^2$



79. The centroid (center of gravity) of a trapezoid may be found by the following

$$x = \frac{b}{3}\,\frac{m + 2n}{m + n}$$



Find x if $m = 2 \quad n = 4 \quad b = 5$

Trial and Error

80. $x^4 - 5x^2 + 4 = 0$

81. $x^2 - 3x + 2 = 0$

82. $\frac{x+1}{2} + \frac{x+3}{4} = 5$

83 $\frac{8x+5}{14} + \frac{7x-3}{6x+2} = \frac{4x+6}{7}$

84. $32 = \frac{64}{\sqrt{x+1}}$

Graphical Solution

85. $x^2 + x - \frac{11}{4} = 0$
86. $x^2 + 4x + 4 = 0$ (Multiple Roots)
87. $x^3 - 6x = 0$
88. $4x^3 - 40x = 0$
89. $3x^4 - 14x + 8 = 0$ (Two roots are imaginary)
90. $r^3 - r^2 + r - 2 = 0$ (Two roots are imaginary)
91. $2x^3 - 15x + 10 = 0$
92. $x^3 - 4x^2 - 5x + 14 = 0$
93. $x^3 - 2x^2 - 7x - 4 = 0$
94. $x^4 - 4x^3 - 4x^2 + 16x + 15 = 0$ (Multiple roots)

Factoring

Factor and solve the following:

95. $x^2 + 2x + 1 = 0$
96. $x^2 - 3x - 4 = 0$
97. $x^2 + x - 3 = 0$
98. $4x^2 + 4x = -1$
99. $3y^2 - 5y + 2 = 0$
100. $5t^2 + 19t + 12 = 0$
101. $5t^2 + 12 = 19t$
102. $2m^3 + 5m^2 - 4m - 3 = 0$

Expand the following:

103. $(x - 1)(2x + 3)$
104. $(2n + 3)(2n - 1)$
105. $(y + 5)(5y + 3)$
106. $(x + 3)(2x + 6)(4x - 5)$
107. $(3x + 4)^2$
108. $(x + 5)^3$
109. $(\sqrt{3x + 9})(\sqrt{3x + 9})$
110. $(x - \sqrt{2x + 3})^2$
111. $(x + \sqrt{x^2 + 4x + 3})^2$
112. $\dfrac{2(3x^3 - x^2 - 12x + 4)}{2x + 4}$
113. $\dfrac{10x^4 + 19x^3 - 6x^2 - 12x + 9}{2x + 3}$
114. $\dfrac{7x^3 - 4x^2 + 6x - 9}{x^2 - 3x + 2}$
115. $\dfrac{3x^3 + 4x^2 - 6x - 8}{x^2 - 2}$
116. $\left(a^2 - \dfrac{2a}{3} + \dfrac{1}{4}\right)\left(2a^2 - \dfrac{a}{3} - \dfrac{1}{2}\right)$
117. $\left(\dfrac{b^2}{3} - \dfrac{b}{2} - \dfrac{1}{5}\right)\left(\dfrac{b^2}{6} - \dfrac{2b}{12} - \dfrac{5}{25}\right)$

118. The product obtained by multiplying the sum of two integers by the difference between the same integers is 27. If the larger number is twice as large as the smaller number, what are the two numbers? Let x represent the smaller number.

119. The area between the circumference of two circles, one within the other, is given by the formula:

$$A = \pi (R_1 + R_2)(R_1 - R_2)$$

Where R_1 = Radius of outer circle
R_2 = Radius of inner circle

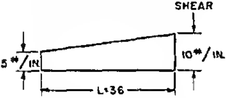
The circles need not be concentric, that is, drawn from the same center. Find the area between the circumference of two circles having diameters of 4 inches and 3 inches.

Completing the Square

120 $x^2 + 10x - 39 = 0$
121. $x^2 - 5x + 6 = 0$
122. $4x^2 + 4x - 54 = 45$
123 $6x^2 + 5x = 1$
124 $7x - 6 - 2x^2 = 0$
125. $3x^2 + 121 = 44x$
126 $4x^2 - 4x = 7$
127. $3x^2 + 9x - 4 = 0$
128. $9x^2 + 6x - 17 = 0$
129. $6x^2 - x = 12$

Quadratic Equations

130. The bending moment is a maximum at the location along the span of a beam where the shear is zero The point, x, of zero shear for a simple beam load as shown is.

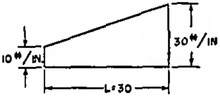


$$120 = 5x + \frac{10-5}{36}\,\frac{x^2}{2} \quad \text{or}$$

$$x^2 + 72x - 1728 = 0$$

Find the point of zero shear.

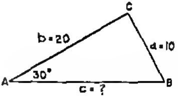
131 Find the point x of zero shear from the equation·



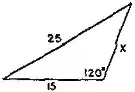
$$250 = 10x + \frac{30-10}{30}\,\frac{x^2}{2} \quad \text{or,}$$

$$x^2 + 30x - 750 = 0$$

132. $a^2 = b^2 + c^2 - 2bc \cos A$
$10^2 = 20^2 + c^2 - (2 \times 20 \times c \times .866)$
Find the true length of side c



133. $25^2 = 15^2 + x^2 - (2)\,(15)\,(x)\,(-.5)$
Find the length of side x



134. The allowable column stress for a steel tube is given by the formula.

$$F_{co} = 135{,}000 - 15.92\,\frac{(L')^2}{\rho} \qquad \text{Short Column}$$

$$F_{co} = \frac{286 \times 10^6}{(L'/\rho)^2} \qquad \text{Long Column}$$

The critical or transition $\dfrac{L'}{\rho}$ is that slenderness ratio above which columns are *long* and below which columns are *short*. At this slenderness

ratio, the allowable stress given by either of the above formulas is the same. A graph of the two equations is tangent at this point; find the critical slenderness ratio L'/ρ for tubes made of alloy steel designed by the above formulas. Also determine the allowablecolumn stress, F_{co} at the critical L'/ρ.

135. The tear-out strength (P_s) of a riveted joint is given by the formula:

$$P_s = 2 \times T.O. \times t \times F_s$$

and the bearing strength (P_{br}) is given by:

$$P_{br} = D \times t \times F_{br}$$

Where $T.O.$ = distance from edge of hole to edge of sheet
Fs = allowable shear stress of sheet
t = thickness of sheet
D = diameter of rivet
F_{br} = allowable bearing stress of sheet

Find the minimum tear-out distance for a riveted joint so that the tear-out strength and bearing strength are equal.

Assume $a = .064$
$F_s = 34{,}000$
$D = \dfrac{3}{16}$
$F_{br} = 82{,}000$

Quadratic Equations (Formula)

136. $x^2 + 4x + 1 = 0$
137. $5x^2 + 10x + 15 = 30$
138. $x^2 + 4x = 2x + 3x^2 - 8$
139. $65x^2 + 96x + 36 = 256x^2$
140. $x^2 + \left(\dfrac{5-x}{3}\right)^2 = 4$
141. $\dfrac{24}{x+9} + \dfrac{24}{x-9} = 6$
142. $x^2 - 0.3x + 1.8 = 0$
143. $x^2 + .25x = 17$
144. $x^2 - 1.25x - 1 = 0$
145. $3x^2 - 0.1x - 0.33 = 0$

Radical Equations

146. $x + \sqrt{x-5} = 11$
147. $2\sqrt{x+1} + x = 7$
147. $\sqrt{2x+29} - \sqrt{x+6} = 3$
149. $\sqrt{x-1} = 3 + \sqrt{x-10}$
150. $x + 5 - 2\sqrt{x+5} + 1 = 3x + 4$
151. $3x + 5 = 2 + \sqrt{3x+4}$
152. $\sqrt{3x-2} - \sqrt{x+3} = 1$
153. $\sqrt{2x+5} - \sqrt{x-6} = 3$
154. $\sqrt{x+6} - 1 = \sqrt{3x+7}$
155. $\sqrt{x+16} - 3 = \sqrt{x-5}$

Simultaneous Equations

Graphical.

156. $x + y = 4$
$x - y = 2$

157. $x + 2y = 3$
$x - 2y = 2$

158. $2y - x = 6$
$2x + y = 8$

159. $x - 5y + 4 = 0$
$3x - 10y + 2 = 0$

160 $$\frac{x}{3}+y=10$$
$$y+\frac{x}{5}=y-3$$

161. $$\begin{aligned} 2x+y&=1\\ 4x+y^2&=17 \end{aligned}$$

162. $$\begin{aligned} 4x^2-y^2&=15\\ 2x-y&=3 \end{aligned}$$

163. $$\begin{aligned} 2x+y&=1\\ x^2+y^2&=1 \end{aligned}$$

164. $$\begin{aligned} 9x^2-4y^2&=35\\ 3x+2y&=7 \end{aligned}$$

165. $$\begin{aligned} 5x^2+10y&=85\\ 2x^2-2y&=10 \end{aligned}$$

Substitution

166. $$\begin{aligned} 5x+2y&=4\\ 7x-3y&=23 \end{aligned}$$

167. $$\begin{aligned} 4x^2-y^2&=15\\ 2x-y&=3 \end{aligned}$$

168. $$\begin{aligned} x+y&=4000\\ .06x+.05y&=230 \end{aligned}$$

169 $$\begin{aligned} .5A+866B&=100\\ 866A-58B&=0 \end{aligned}$$

170. $$\begin{aligned} y^2-x^2&=20\\ x&=\frac{2}{3}y \end{aligned}$$

171. $$\begin{aligned} x^2+xy+y^2&=30\\ x+y&=2 \end{aligned}$$

172. $$\begin{aligned} x^2+y^2&=25\\ x+2y^2&=34 \end{aligned}$$

173. $$\frac{1}{x}+\frac{1}{y}=5$$
$$\frac{1}{x}-\frac{1}{y}=3$$

174. $$\begin{aligned} 2x+3y-4z&=-1\\ x-6y+2z&=3\\ 4x-3y+8z&=5 \end{aligned}$$

175. $$\begin{aligned} 2x-4y-6z&=-6\\ 3x+5y+4z&=16\\ -x+2y+z&=0 \end{aligned}$$

Addition or Subtraction.

176. $$\begin{aligned} 5x+4y&=22\\ 3x+y&=9 \end{aligned}$$

177. $$\begin{aligned} R_1+R_2-50&=0\\ 10R_1-10R_2+250&=0 \end{aligned}$$

178. $$\begin{aligned} x^2+y^2&=5\\ x^2-y^2&=3 \end{aligned}$$

179. $$\frac{10}{x}-\frac{9}{y}=2$$
$$\frac{8}{x}-\frac{15}{y}=-1$$

180. $$\frac{3x}{2}-\frac{4y}{3}=-1$$
$$\frac{2x}{3}-\frac{y}{4}=\frac{7}{12}$$

181. $$\begin{aligned} BX+Y&=B\\ AX+2Y&=2A \end{aligned}$$

182. $$\begin{aligned} .866T-.5c&=100\\ 1.732T+3.0c&=0 \end{aligned}$$

183. $$\begin{aligned} x+y&=23\\ y+z&=25\\ z+x&=24 \end{aligned}$$

184. $$\begin{aligned} 2x+3y&=4z\\ 3x-4y&=5z+4\\ 5x-3z&=y-2 \end{aligned}$$

185. $$\begin{aligned} 2x-y+z&=5\\ 3x+2y+3z&=7\\ 4x-3y-5z&=-3 \end{aligned}$$

Comparison.

186. $2x - 4y = 20$
$\underline{4x - 2y = 16}$

187. $2x + 4 = 6$
$\underline{x^2 - y = 1}$

188. $s - 44T = 0$
$\underline{s = 4T^2}$

189. $x^2 - y^2 = 25$
$\underline{3x = 16y}$

190. $5x + y = 6$
$\underline{2(x + .5)^2 = y^2 - 34}$

191. $y = .577(100 - x)$
$\underline{y = 1.1917x}$

192. $81 = (12 - x)^2 + y^2$
$\underline{36 = x^2 + y^2}$

193. $x^2 - 2y - 1 = y$
$\underline{x - y + 3 = 0}$

194. $y = \dfrac{8}{(x - 2)(x - 1)}$
$\underline{y(x - 3)(x - 1) = 7}$

195. $.25A - .2B = 6$
$\underline{3A + 4B = 8}$

Three Variables.

196. $2x + y - z = 1$
$x + y + z = 6$
$\underline{x + 2y - z = 0}$

197. $4x - 3y + 2z = 1.5$
$x - 6y + 4z = -.5$
$\underline{3x - 2y - z = \frac{7}{12}}$

198. $2x - 3y - z = -4$
$3x + y + 2z = 7$
$\underline{4x - 2y + 2z = -1}$

199. $2x + 3y - 4z = -1$
$x - 6y + 2z = 3$
$\underline{4x - 3y + 8z = 5}$

200. $2x + 7y = 48$
$5y - 2x = 24$
$\underline{x + y + z = 10}$

Division.

201. $x^2 + xy = 20$
$\underline{x + y = 10}$

202. $A^2 + AB = 45$
$\underline{A + B = 9}$

203. $S = 4T^2$
$\underline{S - 44T = 0}$

204. $y = x^2 - 2x - 1$
$\underline{y = x + 3}$

205. $L \sin \theta = \dfrac{WV^2}{GR}$
$\underline{L \cos \theta = W}$

$\operatorname{Tan} \theta = ?$ Note $\operatorname{Tan} \theta = \dfrac{\sin \theta}{\cos \theta}$

206. The sum of two angles of a triangle is 120°. Find the angles if 2/3 of one angle plus 4/5 of the other angle is 90°.

207. The sum of two acute angles is 90°. Find each of the two angles if the larger angle exceeds the smaller by 120°.

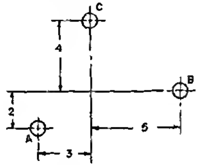
208. The sum of the three interior angles of any triangle is 180°. Find the three angles if the sum of A and B is 90° more than C, and the sum of A and C is 70° more than $2B$.

209. In locating and boring holes in a certain drill jig, it is necessary to know the diameters of three circular holes which are tangent two and two, that is, x is tangent to y and z. The distances between centers are:

$$x \text{ to } y = 3$$
$$x \text{ to } z = 4$$
$$y \text{ to } z = 5$$

Find the diameter of each hole.

210. The location of hole centers is usually shown by the use of rectangular coordinates as in the accompanying diagram. Find the diameter of holes to be drilled at these points if the circles are to be tangent two and two, that is, A is tangent to B and C.



Ratio, Proportion and Variation

Percentage

211 The chemical composition of 24S aluminum alloy is 4.2% copper, 1.5% manganese, 0.5% magnesian, 93.8% aluminum. The density of the material is 173 pounds per cubic foot. Find the number of pounds of each element in one cubic foot of this aluminum alloy.

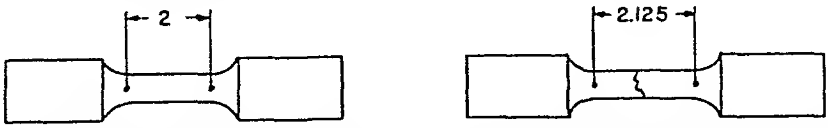
212 The chemical composition of 17S aluminum is 4% copper, 0.5% manganese, 0.5% magnesium and 95% aluminum. The density of the material is 174 pounds per cubic foot. Find the number of pounds of each element in one cubic foot of this aluminum alloy.

213. The yield point strength of aluminum alloy stiffeners can be increased by prestretching. Find the amount of prestretching (inches) required to increase the length of a stringer 3½% if the original length of the stringer is 80 inches.

214. The yield point stress of 24ST extruded shapes in tension is 38,000 pounds per square inch. The ultimate tensile stress of the same material is 58,000 pounds per square inch. Find the percentage of the ultimate tensile strength which is usable without any yielding of the material.

215. The pay load of an airplane is that part of the useful load from which revenue is derived. Find the percent of a gross weight of 48,000 pounds which is represented by a pay load of 15,000 pounds.

216. The percentage elongation of a material is the difference in the gauge length before the test specimen is subjected to any stress and after rupture, expressed in percentage of the original gauge length.



Find the percentage elongation of a material which has a distance between gauge points of 2.125 after rupture if the original gauge length was 2 inches.

217. The percentage reduction of area of a material is the difference between the original cross-sectional area and the least cross-sectional area after rupture, expressed as a percentage of the original cross-sectional area. Find the percentage reduction of area of a material which has a minimum diameter of the test specimen of .450 after rupture, if the original diameter of the test specimen was .505.

Ratio.

218. The specific gravity of a substance is the ratio of the weight of the substance compared to the weight of an equal volume of pure water. The density of pure water is 62.4 pounds per cubic foot. Find the specific gravity of the following commonly used aircraft materials:

MATERIALS	DENSITY OF MATERIAL	DENSITY OF WATER	SPECIFIC GRAVITY
SPRUCE	27	62.4	
MAGNESIUM	109		
24 ST	173		
17 ST	174		
STEEL	490		

219. Find the ratio of the weight of steel to the weight of aluminum alloy *24ST*. Use the densities tabulated in the above example.

220. The wing loading of an airplane is the ratio of the gross weight of the airplane to the wing area. Find the wing loading, if the gross weight is 82,500 pounds and the wing area is 3,000 square feet.

221. The power loading of an airplane is the ratio of the gross weight to the engine horsepower. Find the power loading if the gross weight is 56,000 pounds and the engine horsepower is 4,800.

222. The shear strength of an *AN*4 steel aircraft bolt (¼ dia.) is 3,680 pounds and the shear strength of an *AN*8 steel aircraft bolt (½ dia.) is 14,720 pounds. Find the ratio of the shear strengths of these two bolts. Also compare the ratio of the diameters of the bolts with the ratio of the areas of the bolt shanks.

223. Given: 2.54 centimeters = 1 in.

Inches	Cm
30	
25.4	
	58
	20

224. Given Area $= .7854\,D^2$

Dia (Ins.)	Area (sq. in.)
5	
	16
5	
	40

225. Given. 60 M.P.H. Velocity = 88'/Sec. Velocity

M.P.H.	Ft./sec.
100	
	232
40	
	180

226. Given: Pressure (inches of mercury) = .49116 (pounds per sq. in.)

Inches Hg	Pounds (sq. in.)
	14.7
42	
	100
25	

227. Given: 2π Radians $= 360°$ $\pi = 3.1416$

Degrees	Radians
30	
57.3	
	.75
	1.2

228. Given 1 Kilometer = .62 miles or 1 Kilometer = 3280 feet (1 mile = 5280 ft.)

Miles	Kilometers
10	
	25
576	
	3050

Inverse Proportion

229. The rotation velocities of two gears which are meshed together are inversely proportional to the number of teeth on each gear. Find the R.P.M. of a 32-tooth gear that is driven by a 24-tooth gear which is turning at 600 R.P.M.

230. A gear having 24 teeth drives a gear with 30 teeth. Find the required rotational speed for the 24-tooth gear if the 30-tooth gear is to turn at 1000 R.P.M.

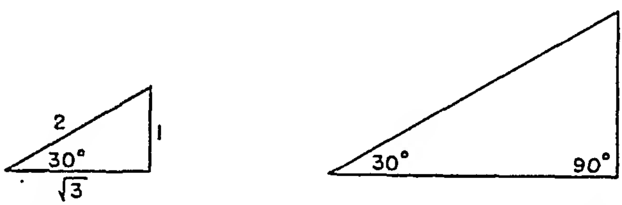
231. The volume of a perfect gas is inversely proportional to the absolute pressure, provided that the temperature remains constant. Find the final volume (V_2) if 50 cubic feet of gas at 15 pounds per square inch pressure is compressed to a pressure of 22.5 pounds per square inch. The temperature is assumed to remain constant.

232. The time required to travel a specified distance varies inversely as the velocity of motion. Find the rate at which an airplane must fly to travel a certain distance in 2 hours and 20 minutes, which can be flown in 3 hours and 30 minutes at a velocity of 120 M.P.H.

233. The landing speed of an airplane is inversely proportional to the square root of the density of the atmosphere. Find the landing speed of an airplane at an altitude of 10,000 feet which normally lands at 60 M.P.H. at sea level. The density of air decreases at an approximate rate of 2% per 1000 feet.

234. The shear strength of a rivet is proportional to the cross-sectional area of the shank. The shear strength of a 3/16 *A17S* rivet is 745 pounds. Find the shear strength of a 1/8 *A17S* rivet.

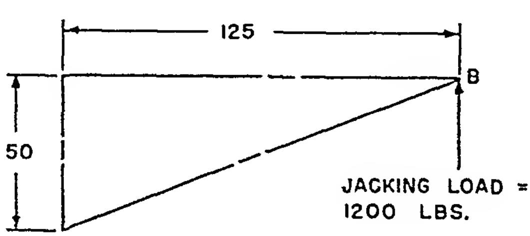
235. The corresponding sides of similar triangles are proportional. Find the increase in altitude in feet when an airplane climbs for 10 seconds at an air speed of 120 M.P.H. The angle of climb is 30°.



236. Interpolation, or finding a value in between two adjacent values given in a table is frequently required. Find the decimal equivalent of 9/64 by the use of data given in the accompanying table.

$\frac{1}{8}$ OR $\frac{8}{64}$	0.125
$\frac{9}{64}$	
$\frac{5}{32}$ OR $\frac{10}{64}$	.15625

237. The axial load in the lift truss of the monoplane wing can be found by proportion by an application of the following principle: The component of the load in a member, taken parellel to any axis, is to the load in the member as the length of the member projected on that axis is to the actual length of the member.



Find the axial load in the lift strut, (*AB*) as a result of a vertical jacking load of 1200 pounds applied at the strut attachment point.

Variation

238 If $F = Ma$ Where
$F =$ Force
$M =$ Mass or $\frac{w}{g}$ where
$w =$ weight
$g = 32\,2$
$a =$ acceleration (Feet/sec/sec)

Find the force necessary to accelerate a 100 pound object 3 feet per sec. per sec; 10 feet per sec. per sec; 100 feet per sec. per sec.

239. The equation for the aspect ratio of an airplane wing is $AR = \frac{b^2}{s}$

Where $AR =$ aspect ratio
$b =$ wing span
$s =$ wing area

If the desired aspect ratio is 10 and the wing area is 2000 square feet, find the span The area remaining the same, find the span when the aspect ratio equals 8, when it equals 6

240. The equation for calculating the lift from the wing of an airplane is as follows

$$L = C_l \frac{\rho V^2}{2}$$

Where $L =$ lift in pounds
$\rho = \ 002378 =$ a function of the density of air
$S = 200 =$ wing area in square feet
$C_l = \ 5 =$ the lift coefficient of the air foil section selected for the particular wing, usually determined by wind tunnel tests
$V =$ speed of plane in feet per sec
Find L when $V = 100, 150, 200$.

241. The T H P (thrust horsepower) delivered through a propeller $= BHP \times N$
Where B H P = brake horse power of the engine.
$N =$ efficiency of the propeller
Let us assume that the efficiency of a certain propeller installed on an engine of 1200 B H P is 82% Plot the curve of thrust horsepower versus brake horsepower throughout the brake horsepower range of the engine.

242. Let us assume that the brake horsepower of the engine mentioned in the preceding problem varies directly with the R.P.M. (revolutions per minute) and that the engine develops 1200 B H P @ 2,000 R P.M Plot a curve of thrust horsepower versus R.P.M. of the engine.

GEOMETRY — PROBLEMS

Sexamigesimal Measurement

Solve the following:

1. $(25° 13' 14'') + (12° 4' 51'') =$
2. $(2° 33' 43'') + (158° 43' 18'') =$
3. $(23° 3' 4'') - (13° 43' 7'') =$
4. $(88° 43' 12'') - (48° 52' 13'') =$
5. $2(4° 13' 8'') =$
6. $5(33° 28' 48'') =$
7. $3(48° 58' 23'') =$
8. $\dfrac{14° 34' 26''}{2} =$
9. $\dfrac{38° 14' 28''}{3} =$
10. $\dfrac{342° 58' 7''}{8} =$

Natural Measurement

11. The length of an arc of a circle is 17" and the diameter is 34". What is the angle in radians that is subtended by the arc.
12. An angle of π radians is subtended by an arc of a circle with a radius of 20". Find the length of the arc.
13. $\theta = 4$ radians subtended by an 8" arc. Find the diameter of the circle represented by this arc.
14. Prove that a complete circle subtends an angle of 2π radians. Hint: Let perimeter equal α (length of the subtending arc).
15. Solve the following:
    (a) $14° =$ radians
    (b) $90° =$ radians
    (c) $150° =$ radians
    (d) 2 radians $= ?°$
    (e) $.5\,\pi$ radians $= ?°$
    (f) $4\,\pi$ radians $= ?°$
16. The radius of a circle is 20". How many degrees of the circle does a 20" arc subtend.
17. The angle subtended by a 15" arc is equal to 60°. What is the diameter of the circle?
18. Assuming that the diameter of the circle in the preceding problem was not known, find its perimeter.

TRIGONOMETRY — PROBLEMS

Types of Triangles

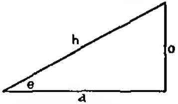
1. Name the two types of triangles.
2. What is a right triangle?

Elements of Triangles

3. Name the six elements of a triangle.
4. What are three ways of determining if a triangle is a right triangle?

Trigonometric Function in Right Triangles

5. Define the sine, cosine and tangent of the angle θ in the right triangle shown below.



$\text{Sin}\,\theta =$
$\text{Cos}\,\theta =$
$\text{Tan}\,\theta =$

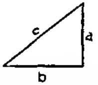
6. State the reciprocal functions for the *Sine, Cosine* and *Tangent.*

FUNCTION	RECIPROCAL FUNCTION
SIN	
COS	
TAN	

7. State the reciprocal functions of problem 6 in terms of a, o, and b as in Problem 5.

Geometric Relations

8 State mathematically the relationships between the three sides on a right triangle



$a = 15 \quad b = ? \quad c = 25$
$a = 15 \quad b = 6 \quad c = ?$
$a = 15 \quad b = 6 \quad c = ?$
$a = 5 \quad b = ? \quad c = 10$

10. If a triangle, similar to that in Problem 9 (a) has a side opposite equal to 12, what will be the hypotenuse?

Use of Tables of Natural Trigonometric Funcitons—Interpolation

11. Find the Sine of 10° 10′ 10′
12. Find the Cosine of 30° 30′ 30″
13. Find the Tangent of 50° 50′ 50″
14 Evaluate $\text{Sin}^{-1}\,.18665$

Solution of Right Triangles

15. Determine the value of the two acute angles in a right triangle with sides equal to 3-4-5.
16. Determine the values of the Sine, Cosine and Tangent for each angle of Problem 15.
17. Give from memory the values of the Sine, Cosine and Tangent of a 30°-60° right triangle.
18. Repeat Problem 17 for a 45° triangle.
19. Using the Cotangent Functions, find the side adjacent of the 30°-60° right triangles with the sides opposite the 30° angles equal to 55, 75 and 95.

20.

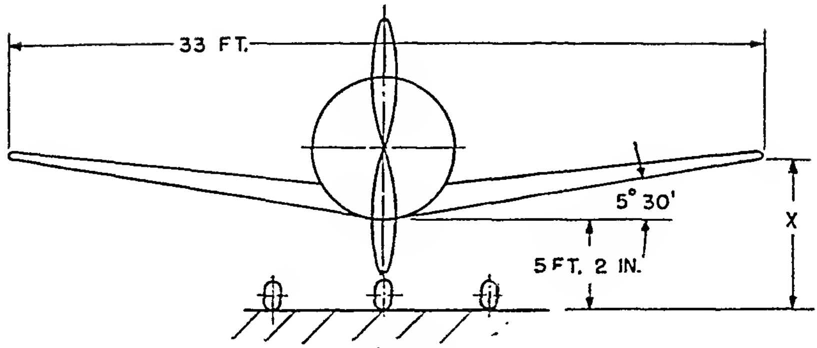


Fig. (a)

If an airplane as that shown in Fig. (a) has a wing spread of 33 feet, a ground clearance of 5 feet 2 inches and a dihedral angle of 5°30′, what is the distance from the ground to the wing tip?

Trigonometric Functions in Oblique Triangles

21. With the center of a circle of unit radius located at the intersection of the ordinate and abscissa line indicate angles of 15°; 108°; 210° and 300°.
22. Indicate which quadrant each angle of Problem 21 is in.
23. Find the Sin, Cos, Tan, Cosec, Sec, and Cotan of each angle of Problem 21. Indicate correct *Sign*—plus or minus.

Geometric Representation of the Trigonometric Functions

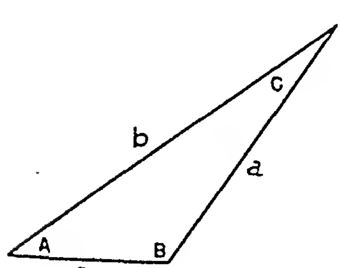
24. By use of a circle with a radius of unity, show diagramatically the values of the Sin, Cos, Tan, Cotan, Sec, and Cosec.

Value of the Functions of Obtuse Angles

25. Find the Sin, Cos, and Tan of 100°, 200° 300° and review the whole problem of functions of such angles.

Oblique Triangles Solved As Right Triangles

26. Solve the following problems:



	A°	B°	C°	a	b	c
a)	28	101	?	?	32	?
b)	21	153	?	57	?	?
c)	64	?	49	?	?	13
d)	?	?	?	39	72	51
e)	?	?	?	10	25	35
f)	16	?	?	?	36	18
g)	?	128	?	99	?	63

Oblique Triangles Solved By Special Formulas

27. Solve and check parts (a), (b), (f) and (g) of Problem 26 by the use of the Special Formulas which are noted under the above heading in the text
28 In reference to Solving of Oblique Triangles
 (a) When can we use the Sine Proportions?
 (b) When can we use the Cosine Law?
 (c) When can we use the Law of Tangents?
29. Review all derivations of the Laws referred to in Problem 28.

Trigonometric Formulas

(Refer to Formulas 218-226 incl)

30 Express the following in terms of Sin ϕ and Cos ϕ only
 (a) $\tan\phi + \sec\phi$ (d) $\cos\phi\tan\phi + \sec\phi$
 (b) $\sin^2\phi + \sec^2\phi$ (e) $\sin\phi\cos\phi\ \tan\phi\cot\phi$
 (c) $\cot^2\phi + \csc^2\phi$
31. Express the following in terms of Sin ϕ only.
 (a) $\sec^2\phi + \tan^2\phi$
 (b) $\sec\phi\tan\phi + \csc\phi$
32 Proof the following identities by transforming the left side only:
 (a) $\sin\phi\cot\phi = \cos\phi$ (d) $\tan\phi + \cot\phi = \sec\phi\ \csc\phi$
 (b) $(1+\tan^2\phi)\sin^2\phi = \tan^2\phi$ (e) $\sin\phi - \tan\phi = \tan\phi\,(\cos\phi - 1)$
 (c) $\dfrac{\sec^2\phi - \tan^2\phi}{\sec^2\phi} = \csc^2\phi$ (f) $\dfrac{\sin\phi - \cos\phi}{\sin\phi + \cos\phi} = \dfrac{\tan\phi - 1}{\tan\phi + 1}$
33 Solve the following by transposing both sides of the equation:
 (a) $\tan\phi + \cot\phi = \sec\phi\csc\phi$
 (b) $\sin\phi\,(1+\tan\phi) - \sec\phi = \csc\phi - \cos\phi\,(1+\cot\phi)$
34 (Refer to formulas 227-234 incl)
 Prove the following:
 (a) $\text{Sin}\,(90° + \phi) = \cos\phi$
 (b) $\cos\,(270° - \phi) = -\sin\phi$
35 By use of *Tables* find approximate values of the following.
 (a) $\sin\,(47° + 32°) - (47° + \sin 32°)$
 (b) $\sin\,(25° - 10°) - (\sin 25° - \sin 10°)$
36. Find the exact values of the following, assuming that ϕ and β are positive acute angles
 (a) $\sin\,(\phi + \beta)$ if $\sin\phi = \dfrac{5}{13}$, $\cos B = \dfrac{4}{5}$
 (b) $\sin\,(\phi - \beta)$ if $\cos\phi = \dfrac{4}{5}$, $\cos\beta = \dfrac{5}{13}$
 (c) $\tan\,(\phi - \beta)$ if $\tan\phi = \dfrac{3}{4}$, $\sin\beta = \dfrac{15}{17}$
37. Solve the following identities
 (a) $\sin\,(60° + \phi) = \dfrac{\sqrt{3}\cos\phi + \sin\phi}{2}$
 (b) $\cos\,(A - B)\cos B - \sin\,(A - B)\sin B = \cos A$
 (c) $\dfrac{\tan\,(x - y) + \tan y}{1 - \tan\,(x - y)\tan y} = \tan x$

38. Prove the formula for $\cos(\phi + \beta)$ directly from a figure in which ϕ and β are positive angles terminating the second quadrant and $\phi + \beta$ an angle terminating the third quadrant.

39. (Refer to formulas 235-244 incl.)
Find the exact values of $\sin 2\phi$; $\cos 2\phi$ and $\tan 2\phi$ in the following:

(a) $\sin \phi = \frac{3}{5}$, if $0° < \phi < 90°$

(b) $\operatorname{ctn} \phi = \frac{4}{3}$, if $360° < \phi < 450°$

(c) $\sin \phi = -\frac{12}{13}$, if $-90° < \phi < 0°$

40. Find the exact values of the functions of the single angles of the following:

(a) $\tan \phi = \frac{12}{5}$, if $180° < \phi < 270°$

(b) $\sin \phi = -\frac{12}{13}$, if $-90° < \phi < 0°$

41. Ptove the following identities:

(a) $1 + \cos 2\phi = 2 \cos^2 \phi$

(b) $\tan \phi = \dfrac{1 - \cos 2\phi}{\sin 2\phi} = \dfrac{\sin 2\phi}{1 + \cos 2\phi}$

(c) $\cos^3 x - \sin^3 x = (\cos x - \sin x)(1 + \frac{1}{2} \sin 2x)$

(d) $\sec 2\phi = 1 + \tan 2\phi \tan \phi$

(e) $\cos 4\phi = 1 - 2 \sin^2 2\phi = 1 - 8 \sin^2 \phi \cos^2 \phi$

(f) $\sin \phi = \dfrac{2 \tan \frac{\phi}{2}}{1 + \tan^2 \frac{\phi}{2}}$

42. (Refer to formulas 245-252 incl.)
Express the following as sums or differences of sines or of cosines.

(a) $2 \sin 2\phi \cos 5\phi$

(b) $\cos \phi \cos 2\phi$

(c) $\sin 5\phi \cos 6\phi$

43. Express the following as products:

(a) $\sin 40° - \sin 50°$

(b) $\cos 20° - \cos 30°$

(c) $\sin 3\phi - \sin \phi$

44. Express as a product involving only tangents and cotangents:

(a) $\dfrac{\sin 40° - \sin 20°}{\sin 40° + \sin 20°}$

(b) $\dfrac{\cos 2\phi - \cos \phi}{\cos 2\phi + \cos \phi}$

45. Prove the following identities:
    (a) $\sin 5x - \sin 3x = 2 \cos 4x \sin x$
    (b) $\cos 8x + \cos 4x = 2 \cos 6x \cos 2x$
    (c) $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan \dfrac{A+B}{2}$
    (d) $\sin x - \cos 2x - \sin 3x = -\cos 2x(2 \sin x + 1)$
46. (Refer to eq 253-255 incl.)
    Solve for the values of angles A, B and C in a triangle with sides:
    $a = 6.20$; $b = 7.00$, $c = 9.10$

Area of Triangles

47. Find the area of the following triangles, using the parts given:
    (a) $b = 64.25$, $A = 24° 23'$, $B = 61° 48'$
    (b) $a = 549.2$; $c = 835$; $A = 41° 8'$
    (c) $a = 1653$, $b = 1777$; $c = 2131$

LOGARITHMS — PROBLEMS

Introduction

1. What mathematical operations can be performed by logarithms?

Definitions and Principles

2. Applying the basic principles of *log* find the values of the following:
   (a) Multiply 3^8 by 3^2
   (b) Multiply 10^2 by 10^3
   (c) Divide 625 by 5^2
   (d) Divide 10^6 by 10^5
   (e) Raise 3^7 to the 6th power or $(3^7)^6 = ?$
   (f) Raise 3^3 to the 3rd power or $(3^3)^3 = ?$
   (g) Extract the square root of 81. or $\sqrt[2]{81} =$
   (h) Express, with a fractional exponent, the cube root of 8 raised to the 7th power.
3. Define a *Logarithm*; and explain how the value of the *base* effects the logarithm.
4. What is a common *log?*
5. If the *log* of 1,000 is equal to 3.0000, which part of the *log* is the characteristic and which part is the mantissa?
6. (a) What part of the logarithm is recorded in *log* tables?
   (b) Are the values positive or negative?
7. Find the *log* of the following:

(a) .0002	(f) 20.
(b) .002	(g) 200.
(c) .02	(h) 2,000
(d) .2	(i) 20,000
(e) 2.0	(Given that: *Log* 2.0 0.30103)

8. Express the following in their equivalent form, such as $-2.301030 = \overline{3}.698970$ and prove mathematically why this is true.
   (a) -2.33968
   (b) -9.84680
   (c) $\overline{4}.82827$
   (d) $\overline{1}.00657$

Rules for Characteristics

9. Review the two methods of determining the characteristic of a *log*.
10. State the characteristic of each log in Problem 7 above.

Augmented Logarithms

11. What are augmented *logs*, and why are they used?
12. Express all the parts of Problem 8 as augmented *logs* and prove mathematically why they are true.
13. Find the values of x, y and z by augmented *logs* and prove by applying the basic principles of *logs*.

Use of Tables of Logarithms

14. Find the *log* of:
    (a) 351
    (b) 59.6
    (c) 9.99
    (d) 1.44
    (e) 0.749
    (f) 0.222
    (g) 0.0064
    (h) 1,022.
    (i) 2,000,000
    (j) 199.9
15. Find the numbers whose *logs* are:
    (a) 3.00217
    (b) 2.87040
    (c) 0.90037
    (d) 7.64048
    (e) $\overline{1}.38021$
    (f) -1.77048
    (g) -6.52699
    (h) $\overline{3}.59040$
    (i) 1.95294
    (j) 2.99917
16. By applying the principles of interpolation, find the *logs* of the following:
    (a) 0.029968
    (b) 2.11370
    (c) 33.00770
    (d) 0.400680
    (e) 1.19230
    (f) 7.76768
    (g) 0.00666
    (h) 2,294,600
    (i) 926,106
    (j) 2.66870
17. Find the numbers whose *logs* are:
    (a) 2.27199
    (b) 1.09081
    (c) $\overline{2}.72333$
    (d) $\overline{4}.83018$
    (e) $\overline{1}.70779$
    (f) 7.65749
    (g) 4.97505
    (h) 0.01498
    (i) 0.30227
    (j) $\overline{1}.47425$

Fundamental Operations Using Logarithms

18. Find the values of the following by *logs*:
    (a) (10) (760)
    (b) (105) (88)
    (c) (8) (967)
    (d) (291) (298)
    (e) (0.77) (0.88)
    (f) (2.88) (7.98)
    (g) (1.14) (2,000,800)
    (h) (0.008) (0.00676)
    (i) (0.186) (22.83)
    (j) (3.43) (6,845)

19 (a) 92/87 (b) 11/21 (c) 0 6/0 47 (d) 2,117/124 (e) 899/62 (f) 100/2,100 (g) 2/606 (h) 0 884/0.821 (i) 1.835/2 647 (j) 1,008,000/67,200

20 (a) 25^4 (b) 6^9 (c) $29\,1^2$ (d) $4^{2\,1}$ (e) 968^3 (f) $8\,8^6$ (g) $7{,}620^2$ (h) $(0\,11)^3$ (i) $(0\,002)^3$ (j) $3\,8^5$

21 (a) $\sqrt{84}$ (b) $\sqrt{6{,}942}$ (c) $\sqrt[3]{1{,}010}$ (d) $\sqrt{3{,}260}$ (e) $\sqrt{0\,1186}$ (f) $\sqrt{272}$ (g) $\sqrt[3]{88}$ (h) $\sqrt[3]{675}$ (i) $\sqrt{7}$ (j) $\sqrt{1000}$

Cologarithms

22 What is a *cologarithm*?

23 Solve all of Problem 19 by the use of *cologs*.

Division or Multiplication of Logarithms

24 Is it true that the difference of the *logs* of two numbers is equal to the *log* of the difference of the two numbers? Prove.

25 Solve for (x) in the following

(a) $(x)(\log 299) = \log 68.4$

(b) $(x)(\log 41) = \log 8{,}960$

(c) $x = (\log 7{,}600)(\log 6)$

(d) $\dfrac{96}{213} = (\tfrac{1}{2})^x$

Solution of Equations Using Logarithms

26 Solve for x

(a) $(9)^{1\,9} = (266)(x)$

(b) $187\,8 = (x)(18)^3$

(c) $(23\,7)^{3\,3} = (22)^{1\,4}(x)$

27. Solve for x

(a) $x = \sqrt{\dfrac{2{,}120}{(100)^2(227.6)}}$ (c) $(624.0)^2(x) = \sqrt[3]{\dfrac{2{,}000{,}000}{(87\,7)(1.022)}}$

(b) $(89.7)(x) = \sqrt[3]{39.86}$ (d) $x = \sqrt{\dfrac{1{,}298}{(400)^3(34.7)}}$

28. Solve for x

(a) $x = \left[\dfrac{3{,}768}{(.877)^3}\right]^{5/3}$

(b) $x = \left[\dfrac{.00876}{.01273}\right]^{2/3}$

(c) $107.6x = [01080]^{012}$

29. Solve for x

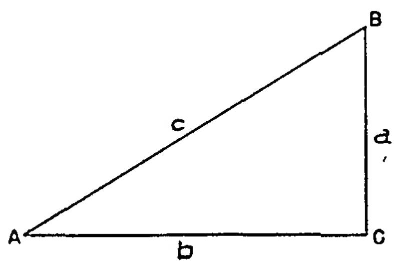
(a) $x = \left[122 - \left(\frac{2}{6.8}\right)^{.8}\right]28.8$

(b) $(3.7)^2 x = \left[8.7 - \left(\frac{1}{4.4}\right)^{.23}\right]37.88$

(c) $x = \left[1 - \left(\frac{1}{11}\right)^{.011}\right]2{,}677$

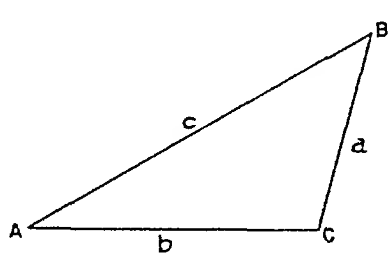
Solution of Triangles Using Logarithms

30. Find the unknowns of the following triangles by natural *logs* and *logs* of trigonometric functions:



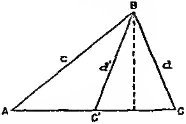
(a) $c = 120$; $A = 31°$
(b) $a = 637$; $A = 4° \, 35'$
(c) $a = 2.189$; $B = 45° \, 25'$
(d) $b = 16.93$; $B = 51° \, 2'$
(e) $a = 0.7183$; $c = 9.914$

31. (a) $A = 47° \, 13'$; $B = 65° \, 24'$; $a = 43.18$
(b) $A = 65° \, 50'$; $B = 38° \, 37'$; $b = 835.6$
(c) $A = 68° \, 41'$; $B = 1° \, 2'$; $c = 9.433$
(d) $A = 61° \, 27'$; $C = 33° \, 22'$; $a = 3541$

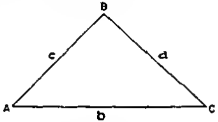


32. (a) $a = 443$; $c = 439$; $A = 40° \, 12'$
(b) $a = 724.7$; $c = 787.5$; $A = 65° \, 15'$

(c) $a = 1149$, $b = 1246$; $A = 67° 16'$
(d) $a = 3541$; $c = 4017$; $A = 61° 27'$



33 (a) $a = 7$, $b = 9$, $C = 48°$ Find c
(b) $b = 100$, $c = 200$, $A = 29° 34'$ Find a
(c) $a = 147.1$, $b = 175.9$, $C = 43° 43'$
(d) $b = 9641$, $c = 8999$, $A = 67° 21'$
(e) $b = 25.25$, $c = 97.46$, $A = 98° 49'$



34 Write the equations for converting common *logs* to natural *logs* and visa versa

35 If the common *logs* of x are as follows, find the natural *log* equivalents:
(a) $Log_{10} X = 0.8186$
(b) $Log_{10} X = 1.8912$
(c) $Log_{10} X = 0.00113$

36 If the natural *logs* are given, find the common *logs*.
(a) $Log_e X = 0.6931$
(b) $Log\ X = 1.7281$
(c) $Log_e X = 1.4310$

Natural or Naperian Logarithms

37 Find the natural *logs* of the following
(a) 0.0097
(b) 1.68270
(c) 102.870
(d) 0.18431
(e) 400,000

ANALYTICAL GEOMETRY OF STRAIGHT LINES — PROBLEMS

Introduction

1. Define (a) Dependent Variable and
(b) Independent Variable

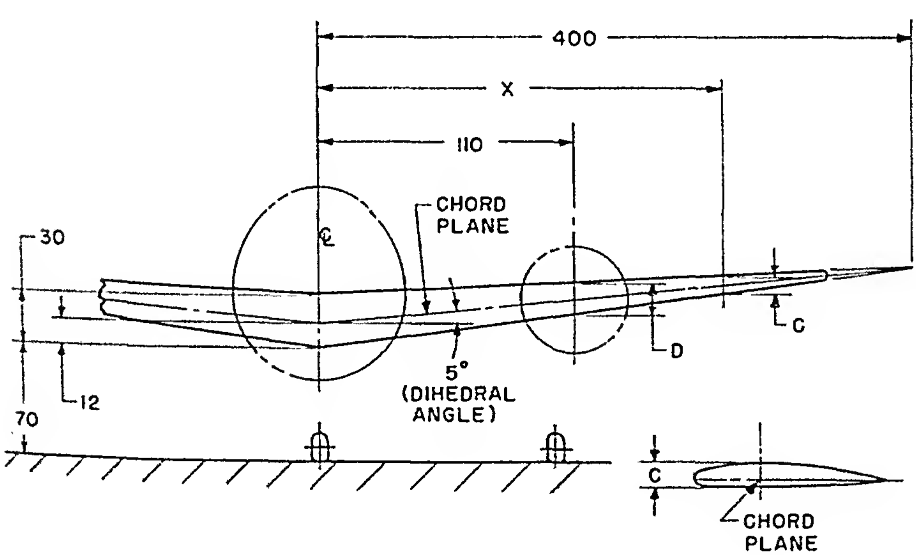
Give examples

Straight Lines

2. What are the distinctive characteristics of the lines $y = 6$; $y = 22$ and $x = 10$.
3. Does the line $y = 99x$ pass through the ordinate? If not what is the intercept?
4. Plot the curve of Problem 3.
5. May the equation of the line of Problem 3 be classed as a linear equation? If so, why?
6. Does the point $x = 13$; $y = 1287$; lie on the curve of Problem 3? Prove.

Slope of a Straight Line

7. Define positive and negative slopes.
8. When does a line have zero slope?
9. When does a line have no slope?
10. If (x) decreases and (y) also decreases, what is the sign of the slope of the line?
11. If (y) decreases and (x) increases, what is the sign of the slope of the line?
12. What are the axes' intercepts of the following equations:
    (a) $y = 2x + 4$ (c) $3y = -4x + 8$
    (b) $y = 2x - 20$ (d) $4y = 4x + 4$
13. What are the values and sign values of the slopes of the curves of the four curves of Problem 12?
14. From Fig. below determine the equation which expresses the relationship between x and C. Using this equation find the value of D.



Interpretation of Slope Term

15. What kind of lines have their slopes equal?
16. If the slope of a line is equal to the negative reciprocal of that of another, what is the relationship between the two lines?

17 If one number is the negative reciprocal of another, is their product or their quotient equal to minus one?

18 Prove which of the following pairs of lines are parallel, perpendicular, or neither:

(a) $y = 2x + 4$	(c) $y = 12x$
(a_1) $6y = -10 + 12x$	(c_1) $3y = -36x + 7$
(b) $2x = 10 + y$	(d) $14x = 18y - 6$
(b_1) $3y - 6x = 2$	(d_1) $100 + 4y = 3x$

19 Find the angle between the two lines of Problem 18 which are neither parallel nor perpendicular

20 If the slope of one line is equal to -68 and that of a second line equals .82, what is the angle between the two in degrees and minutes?

Equation of Any Straight Line

21 Write the equations of the following lines if the slopes and the intercepts are as follows:

	Slope	Intercept
(a)	$+2$	$x = -2,\ y = 0$
(b)	$+7$	$x = +4;\ y = 0$
(c)	-66	$x = 0,\ y = 0$
(d)	$+1$	$y = -22,\ x = 0$
(e)	-2	$y = -11,\ x = 0$

Such as $y = (m)\ (x) + b$

22 Write the equations of the following lines if the slopes of the lines and a point on each line are as follows

	Slope	Point (P)
(a)	$+9$	$x = 0,\ y = 2$
(b)	-2	$x = 9;\ y = 3$
(c)	$+2$	$x = -2;\ y = +2$
(d)	-4	$x = 12,\ y = 1$
(e)	$+1$	$x = -4;\ y = -6$

23 Is point P_2 ($x = 18$, $y = 6$) on the same line as that of part (b) in Problem 22 above?

Simultaneous Equation of Straight Lines

24 Is it true that a set of lines must intersect at but one point to fulfill the requirements of independent equations?

25. Define inconsistent and also equivalent equations?

26 Which set of lines of Problem 18 are independent, inconsistent or equivalent?

Distance from a Point to a Straight Line

27. What is the shortest distance from a point to a line?

28. Find the shortest distances between the following points and their respective lines:

(a)	Point. $x = 10,\ y = 2,$	Line $4x - 4y + 15 = 0$
(b)	Point. $x = 1,\ y = 20,$	Line $9x + y = 0$
(c)	Point.....$x = 5,\ y = 8,$	Line $3x + 8y = 17$

Area Beneath a Straight Line Segment

29. Find the area under the curve $y = 9x + 10$ which is between the two vertical lines drawn at $x = 1$ and $x = 2$ and the horizontal line $y = 0$.

30. Repeat the above for:

(a) Line......$2x + 9y - 6 = 0$
$x = 0$
$x = 9$
$y = 0$

(b) Line......$7x + 10y + 10 = 0$
$x = 3$
$x = 4$
$y = 2$

APPENDIX—SLIDE RULE—PROBLEMS

Division

1. $8 \div 32$
2. $32 \div 8$
3. $45 \div 82$
4. $82 \div 45$
5. $61 \div 69$
6. $80 \div 15$
7. $15 \div 80$
8. $450 \div 1500$
9. $12.4 \div 13.8$
10. $7.38 \div 3.30$
11. $5.86 \div 2.11$
12. $6.13 \div 4.27$
13. $3.14 \div 10.00$
14. $10.00 \div 3.14$
15. $18.7 \div 0.214$
16. $.00626 \div .00038$
17. $.00158 \div 37.6$
18. $.375 \div .015$
19. $.015 \div .375$
20. $.0475 \div 22.1$

Multiplication

21. 2×1.7
22. 1.3×3.1
23. 30×25
24. 300×25
25. 30×250
26. $3 \times .25$
27. $3 \times .025$
28. $.3 \times 25$
29. $.03 \times .025$
30. 18.00×4.30
31. 1.8×430
32. $.0018 \times 59.6$
33. $.0782 \times .0789$
34. $1{,}010 \times 2.66$

35.

	16	17	18	19	20	21	22	23	24
40									
41									
42									
43									

Combined Multiplication and Division

36. $\dfrac{2 \times 3.6}{4}$

37. $\dfrac{11 \times 16}{3.4}$

38. $\dfrac{6 \times 7.8}{4.63}$

39. $\dfrac{100 \times 89}{2.87}$

40. $\dfrac{11 \times 392}{36}$

41. $\dfrac{8.2}{10.2 \times 17}$

42. $\dfrac{55}{17.8\ \ 2.8}$

43. $\dfrac{8.77 \times 87.2}{76.0\ \ 3.1}$

44. $\dfrac{1.35 \times 4.71}{0.77 \times 8.1}$

45. $\dfrac{2{,}010 \times 399}{834}$

Proportions

Find x for the following:

46. $\frac{8}{1.2} = \frac{x}{7 6}$
47. $\frac{1}{7.8} = \frac{x}{822}$
48 $\frac{x}{5.31} = \frac{70 2}{4 07}$
49 $\frac{287}{x} = \frac{1 88}{90.1}$
50. $\frac{100}{87 3} = \frac{53.4}{x}$
51. $\frac{398}{412} = \frac{2 1}{x}$
52 $\frac{2 85}{4 33} = \frac{x}{8}$
53 $\frac{x}{2} = \frac{8 2}{11 2}$
54 $\frac{x}{12.4} = \frac{7.77}{87.6}$
55. $\frac{33.3}{x} = \frac{5.30}{11.8}$
56 $\frac{.0068}{2.63} = \frac{x}{.0194}$

Square Roots and Squares of Numbers

Find the square root of the following

57. .000218
58 .218
59 21 8
60 8.77
61 43 7
62 0187
63 00290
64 9 29
65 834,000
66. 6,620
67. 376

Find the squares of the following

68 3 11
69. 8 96
70 100
71. 62 1
72 1 78
73 3 99
74 11 7
75 462
76 00111
77. .285
78 .000722
79 1020
80 49.9
81. 773
82 5 43

Square Root of the Sum or the Difference of Two Squares

83 $\sqrt{30^2 + 120^2}$
84 $\sqrt{45^2 + 81^2}$
85 $\sqrt{9^2 + 101^2}$
86 $\sqrt{80^2 - 22^2}$
87. $\sqrt{100^2 - 10^2}$

Cube Roots and Cubes of Numbers

Find the cube roots of the following

88 2 0
89 11 8
90 70 7
91. 2010
92 343
93 21 7
94 0178
95. .0000229
96. 67 10
97 100 0

Find the cubes of the following

98 11.1
99 31.0
100. 47.0
101. 48 0
102. 50 0
103 81 3
104 2.89
105 100 0
106. 1 99
107 8 77

Trigonometric Functions

Find the Sine of the following:

108. 90°
109. 45°
110. 41°
111. 10° 32′
112. 1° 6′
113. 8° 31′
114 99° 30′
115 10′
116 71° 30′
117 160°

Find the Cosine of the following:

118. 80°
119. 75°
120. 13°
121. $1^\circ\,2'$
122. $54^\circ\,15'$
123. $20^\circ\,30'$
124. $87^\circ\,45'$

By the use of the slide rule find the *logs* of the following:

125. 1.44
126. 53.4
127. 0.317
128. 0.0722
129. 443.0

Logarithms

By the use of the *log* scales find the value of the following:

130. $(3.3)^4$
131. $(99)^5$
132. $(10.2)^{10.1}$
133. $\sqrt{8.76}$
134. $\sqrt[3]{213}$
135. $\sqrt[1.2]{43.4}$

TRIGONOMETRIC FUNCTIONS

0° READ DOWN 1°

0°

′	sin	tan	cot	cos	
0	.00000	.00000	∞	1.0000	60
1	029	029	3437.7	000	59
2	058	958	1718.9	000	58
3	087	987	1145.9	000	57
4	116	116	859 44	000	56
5	145	145	687.55	000	55
6	175	175	572.96	000	54
7	204	201	491.11	000	53
8	00233	.00233	429 72	000	52
9	262	262	381.97	000	51
10	291	291	343.77	1.0000	50
11	320	320	312.52	.99999	49
12	349	349	286.48	999	48
13	378	378	264.44	999	47
14	407	407	245.55	999	46
15	436	436	229 18	999	45
16	465	465	214.86	999	44
17	.00495	.00495	202.22	999	43
18	524	521	190 98	999	42
19	553	553	180 93	998	41
20	582	582	171 89	.99998	40
21	611	911	163.70	998	39
22	640	640	156.26	998	38
23	669	660	149.47	998	37
24	998	698	143.24	998	36
25	.00727	.00727	137.51	997	35
26	756	756	132.22	997	34
27	785	785	127.32	997	33
28	814	815	122.77	997	32
29	844	844	118.54	996	31
30	873	873	114.59	.99996	30
31	902	902	110.89	996	29
32	931	931	107.43	996	28
33	960	960	104.17	995	27
34	.00989	.00989	101 11	995	26
35	.01018	.01018	98.218	995	25
36	047	047	95.489	995	24
37	076	076	92.908	994	23
38	105	105	90.463	994	22
39	134	135	88.144	994	21
40	164	164	85.940	.99993	20
41	193	193	83.844	993	19
42	222	222	81.847	993	18
43	.01251	251	79 943	992	17
44	280	.91280	78.126	992	16
45	309	309	76.390	991	15
46	338	338	74.729	991	14
47	367	367	73.139	991	13
48	396	396	71.015	990	12
49	425	425	70.153	990	11
50	454	455	68.750	.99989	10
51	483	484	67.402	989	9
52	.01513	.01513	66.105	989	8
53	542	542	64.858	988	7
54	571	571	63.657	988	6
55	600	600	62.499	987	5
56	629	629	61 383	987	4
57	658	658	60.306	986	3
58	687	687	59 266	986	2
59	716	716	58.261	985	1
60	.01745	.01746	57.290	.99985	0
	cos	cot	tan	sin	′

1°

′	sin	tan	cot	cos	
0	.01745	.01746	57.290	.99985	60
1	774	775	56.351	984	59
2	803	804	55.442	984	58
3	832	833	54.561	983	57
4	862	862	53.709	983	56
5	891	891	52.882	982	55
6	920	920	52.081	982	54
7	949	949	51.303	981	53
8	.01978	.01978	50.549	980	52
9	.02007	.02007	49 816	980	51
10	036	036	49.104	.99979	50
11	065	066	48.412	979	49
12	094	095	47.740	978	48
13	123	124	47.085	977	47
14	152	153	46.449	977	46
15	181	182	45.829	976	45
16	211	211	45.226	976	44
17	240	.02240	44.639	975	43
18	269	269	44.066	974	42
19	298	298	43.508	974	41
20	.02327	328	42.964	.99973	40
21	356	357	42.433	972	39
22	385	386	41.916	972	38
23	414	415	41.411	971	37
24	443	444	40 917	070	36
25	472	473	40.436	969	35
26	501	.02502	39.965	969	34
27	530	531	39.506	968	33
28	560	560	39.057	967	32
29	589	589	38.618	966	31
30	.02618	619	38.188	.99966	30
31	647	648	37.769	965	29
32	676	677	37.358	964	28
33	705	706	36.956	963	27
34	734	735	36.563	963	26
35	763	.92764	36.178	962	25
36	792	793	35 801	961	24
37	.02821	822	35.431	960	23
38	850	851	35.070	959	22
39	879	881	34.715	959	21
40	908	910	34.368	.99958	20
41	938	939	34.027	957	19
42	967	968	33.694	956	18
43	.02996	.02997	33.366	955	17
44	.03025	.03026	33.045	954	16
45	054	055	32.730	953	15
46	083	084	32.421	952	14
47	112	114	32.118	952	13
48	141	143	[illegible]	[illegible]	12
49	170	172	31.528	950	11
50	199	201	31.242	.99949	10
51	228	230	30.960	948	9
52	.03257	.03259	30.683	947	8
53	286	288	30.412	946	7
54	316	317	30 145	945	6
55	345	346	29 882	944	5
56	374	376	29.624	943	4
57	403	405	29.371	942	3
58	432	434	29.122	941	2
59	461	463	28.877	940	1
60	.03490	.03492	28.636	.99939	0
	cos	cot	tan	sin	′

89° READ UP 88°

2° READ DOWN 3°

′	sin	tan	cot	cos	
0	.03490	.03492	28.636	.99939	60
1	519	521	.399	938	59
2	548	550	28.166	937	58
3	577	579	27.937	936	57
4	606	609	.712	935	56
5	635	638	.490	934	55
6	664	667	.271	933	54
7	693	696	27.057	932	53
8	723	.03725	26.845	931	52
9	.03752	754	.637	930	51
10	781	783	.432	.99929	50
11	810	812	.230	927	49
12	839	842	26.031	926	48
13	868	871	25.835	925	47
14	897	900	.642	924	46
15	926	929	.452	923	45
16	955	958	.264	922	44
17	.03984	.03987	25.080	921	43
18	.04013	.04016	24.898	919	42
19	042	046	.719	918	41
20	071	075	.542	.99917	40
21	100	104	.368	916	39
22	129	133	.196	915	38
23	159	162	24.026	913	37
24	188	191	23.859	912	36
25	217	220	.695	911	35
26	246	.04250	.532	910	34
27	.04275	279	.372	909	33
28	304	308	.214	907	32
29	333	337	23.058	906	31
30	362	366	22.904	.99905	30
31	391	395	.752	904	29
32	420	424	.602	902	28
33	449	454	.454	901	27
34	478	.04483	.308	900	26
35	.04507	512	.164	898	25
36	536	541	22.022	897	24
37	565	570	21.881	896	23
38	594	599	.743	894	22
39	623	628	.606	893	21
40	653	658	.470	.99892	20
41	682	687	.337	890	19
42	.04711	.04716	.205	889	18
43	740	745	21.075	888	17
44	769	774	20.946	886	16
45	798	803	.819	885	15
46	827	833	.693	883	14
47	856	862	.569	882	13
48	885	891	.446	881	12
49	914	920	.325	879	11
50	943	949	.206	.99878	10
51	.04972	.04978	20.087	876	9
52	.05001	.05007	19.970	875	8
53	030	037	.855	873	7
54	059	066	.740	872	6
55	088	095	.627	870	5
56	117	124	.516	869	4
57	146	153	.405	867	3
58	175	182	.296	866	2
59	205	212	.188	864	1
60	.05234	.05241	19.081	.99863	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.05234	.05241	19.081	.99863	60
1	263	270	18.976	861	59
2	292	299	.871	860	58
3	321	328	.768	858	57
4	350	357	.666	857	56
5	379	387	.564	855	55
6	408	416	.464	854	54
7	437	445	.366	852	53
8	466	474	.268	851	52
9	.05495	.05503	.171	849	51
10	524	533	18.075	.99847	50
11	553	562	17.980	846	49
12	582	591	.886	844	48
13	611	620	.793	842	47
14	640	649	.702	841	46
15	669	678	.611	839	45
16	698	708	.521	838	44
17	.05727	737	.431	836	43
18	756	.05766	.343	834	42
19	785	795	.256	833	41
20	814	824	.169	.99831	40
21	844	854	17.084	829	39
22	873	883	16.999	827	38
23	902	912	.915	826	37
24	931	941	.832	824	36
25	960	970	.750	822	35
26	.05989	.05999	.668	821	34
27	.06018	.06029	.587	819	33
28	047	058	16.507	817	32
29	076	087	.428	815	31
30	105	116	.350	.99813	30
31	134	145	.272	812	29
32	163	175	.195	810	28
33	192	204	.119	808	27
34	221	233	16.043	806	26
35	.06250	262	15.969	804	25
36	279	.06291	.895	803	24
37	308	321	.821	801	23
38	337	350	.748	799	22
39	366	379	.676	797	21
40	395	408	.605	.99795	20
41	424	438	.534	793	19
42	453	467	15.464	792	18
43	.06482	496	.394	790	17
44	511	.06525	.325	788	16
45	540	554	.257	786	15
46	569	584	.189	784	14
47	598	613	.122	782	13
48	627	642	15.056	780	12
49	656	671	14.990	778	11
50	685	700	.924	.99776	10
51	.06714	730	.860	774	9
52	743	.06759	.795	772	8
53	773	788	.732	770	7
54	802	817	.669	768	6
55	831	847	.606	766	5
56	860	876	.544	764	4
57	889	905	.482	762	3
58	918	934	.421	760	2
59	947	963	.361	758	1
60	.06976	.06993	14.301	.99756	0
	cos	cot	tan	sin	′

87° READ UP 86°

4° READ DOWN

′	sin	tan	cot	cos	
0	.06976	.06993	14.301	.99756	60
1	.07005	.07022	.241	754	59
2	034	051	.182	752	58
3	063	080	.124	750	57
4	092	110	.065	748	56
5	121	139	14.008	746	55
6	150	168	13.951	744	54
7	179	197	.894	742	53
8	208	227	.838	740	52
9	237	.07256	.782	738	51
10	.07266	285	.727	.99736	50
11	295	314	.672	734	49
12	324	344	.617	731	48
13	353	373	.563	729	47
14	382	402	13.510	727	46
15	411	431	.457	725	45
16	440	461	.404	723	44
17	469	490	.352	721	43
18	.07498	.07519	.300	719	42
19	527	548	.248	716	41
20	556	578	.197	.99714	40
21	585	607	.146	712	39
22	614	636	.096	710	38
23	643	665	13.046	708	37
24	672	695	12.996	705	36
25	701	724	.947	703	35
26	.07730	.07753	.898	701	34
27	759	782	.850	699	33
28	788	812	.801	696	32
29	817	841	.754	694	31
30	846	870	.706	.99692	30
31	875	899	.659	689	29
32	904	929	12.612	687	28
33	933	958	.566	685	27
34	962	.07987	.520	683	26
35	.07991	.08017	.474	680	25
36	.08020	046	.429	678	24
37	049	075	.384	676	23
38	078	104	12.338	673	22
39	107	134	.295	671	21
40	136	163	.251	.99668	20
41	165	192	.207	666	19
42	194	221	.163	664	18
43	223	.08251	.120	661	17
44	.08252	280	.077	659	16
45	281	309	12.035	657	15
46	310	339	11.992	654	14
47	339	368	.950	652	13
48	368	397	.909	649	12
49	397	427	.867	647	11
50	426	456	.826	.99644	10
51	455	485	.785	642	9
52	.08484	.08514	.745	639	8
53	513	544	.705	637	7
54	542	573	11.664	635	6
55	571	602	.625	632	5
56	600	632	.585	630	4
57	629	661	.546	627	3
58	658	690	.507	625	2
59	687	720	.468	622	1
60	.08716	.08749	11.430	.99619	0
	cos	cot	tan	sin	′

85° READ UP

5°

′	sin	tan	cot	cos	
0	.08716	.08749	11.430	.99619	60
1	745	778	.392	617	59
2	774	807	.354	614	58
3	803	837	.316	612	57
4	831	866	.279	609	56
5	860	895	.242	607	55
6	889	925	.205	604	54
7	918	954	.168	602	53
8	947	.08983	.132	599	52
9	.08976	.09013	.095	596	51
10	.09005	042	.059	.99594	50
11	034	071	11.024	591	49
12	063	101	10.988	588	48
13	092	130	.953	586	47
14	121	159	.918	583	46
15	150	189	.883	580	45
16	179	218	.848	578	44
17	208	.09247	.814	575	43
18	.09237	277	.780	572	42
19	266	306	.746	570	41
20	295	335	.712	.99567	40
21	324	365	.678	564	39
22	353	394	10.645	562	38
23	382	423	.612	559	37
24	411	453	.579	556	36
25	440	.09482	.546	553	35
26	.09469	511	.514	551	34
27	498	541	.481	548	33
28	527	570	.449	545	32
29	556	600	.417	542	31
30	585	629	.385	.99540	30
31	614	658	.354	537	29
32	642	688	10.322	534	28
33	671	.09717	.291	531	27
34	700	746	.260	528	26
35	.09729	776	.229	526	25
36	758	805	.199	523	24
37	787	834	.168	520	23
38	816	864	.138	517	22
39	845	893	.108	514	21
40	874	923	.078	.99511	20
41	903	952	.048	508	19
42	932	.09981	10.019	506	18
43	961	.10011	9.9893	503	17
44	.09990	040	.9601	500	16
45	.10019	069	.9310	497	15
46	048	099	.9021	494	14
47	077	128	.8734	491	13
48	106	158	.8448	488	12
49	135	187	.8164	485	11
50	164	216	.7882	.99482	10
51	192	.10246	.7601	479	9
52	221	275	9.7322	476	8
53	.10250	305	.7044	473	7
54	279	334	.6768	470	6
55	308	363	.6493	467	5
56	337	393	.6220	464	4
57	366	422	.5949	461	3
58	395	452	.5679	458	2
59	424	481	.5411	455	1
60	.10453	.10510	9.5144	.99452	0
	cos	cot	tan	sin	′

84°

6° READ DOWN

′	sin	tan	cot	cos	
0	.10453	.10510	9.5144	.99452	60
1	482	540	.4878	449	59
2	511	569	.4614	446	58
3	540	599	.4352	443	57
4	569	628	.4090	440	56
5	597	657	.3831	437	55
6	626	687	.3572	434	54
7	655	716	.3315	431	53
8	684	.10746	.3060	428	52
9	.10713	775	.2806	424	51
10	742	805	9.2553	.99421	50
11	771	834	.2302	418	49
12	800	863	.2052	415	48
13	829	893	.1803	412	47
14	858	922	.1555	409	46
15	887	952	.1309	406	45
16	916	.10981	.1065	402	44
17	945	.11011	.0821	399	43
18	.10973	040	.0579	396	42
19	.11002	070	.0338	393	41
20	031	099	9.0098	.99390	40
21	060	128	8.9860	386	39
22	089	158	.9623	383	38
23	118	187	.9387	380	37
24	147	217	.9152	377	36
25	176	.11246	.8919	374	35
26	205	276	.8686	370	34
27	234	305	.8455	367	33
28	.11263	335	.8225	364	32
29	291	364	.7996	360	31
30	320	394	.7769	.99357	30
31	349	423	8.7542	354	29
32	378	452	.7317	351	28
33	407	.11482	.7093	347	27
34	436	511	.6870	344	26
35	465	541	.6648	341	25
36	494	570	.6427	337	24
37	.11523	600	.6208	334	23
38	552	629	.5989	331	22
39	580	659	.5772	327	21
40	609	688	.5555	.99324	20
41	638	.11718	8.5340	320	19
42	667	747	.5126	317	18
43	696	777	.4913	314	17
44	725	806	.4701	310	16
45	.11754	836	.4490	307	15
46	783	865	.4280	303	14
47	812	895	.4071	300	13
48	840	924	.3863	297	12
49	869	954	.3656	293	11
50	898	.11983	.3450	.99290	10
51	927	.12013	8.3245	286	9
52	956	042	.3041	283	8
53	.11985	072	.2838	279	7
54	.12014	101	.2636	276	6
55	043	131	.2434	272	5
56	071	160	.2234	269	4
57	100	190	.2035	265	3
58	129	219	.1837	262	2
59	158	249	.1640	258	1
60	.12187	.12278	8.1443	.99255	0
	cos	cot	tan	sin	′

83° READ UP

7° READ DOWN

′	sin	tan	cot	cos	
0	.12187	.12278	8.1443	.99255	60
1	216	308	.1248	251	59
2	245	338	.1054	248	58
3	274	367	.0860	244	57
4	302	397	.0667	240	56
5	331	426	.0476	237	55
6	360	456	.0285	233	54
7	389	485	8.0095	230	53
8	418	.12515	7.9906	226	52
9	.12447	544	.9718	222	51
10	476	574	.9530	.99219	50
11	504	603	.9344	215	49
12	533	633	.9158	211	48
13	562	662	.8973	208	47
14	591	692	.8789	204	46
15	620	722	.8606	200	45
16	649	.12751	7.8424	197	44
17	678	781	.8243	193	43
18	.12706	810	.8062	189	42
19	735	840	.7882	186	41
20	764	869	.7704	.99182	40
21	793	899	.7525	178	39
22	822	929	.7348	175	38
23	851	958	.7171	171	37
24	880	.12988	.6996	167	36
25	908	.13017	7.6821	163	35
26	937	047	.6647	160	34
27	966	076	.6473	156	33
28	.12995	106	.6301	152	32
29	.13024	136	.6129	148	31
30	053	165	.5958	.99144	30
31	081	195	.5787	141	29
32	110	224	.5618	137	28
33	139	.13254	.5449	133	27
34	168	284	7.5281	129	26
35	197	313	.5113	125	25
36	.13226	343	.4947	122	24
37	254	372	.4781	118	23
38	283	402	.4615	114	22
39	312	432	.4451	110	21
40	341	461	.4287	.99106	20
41	370	.13491	.4124	102	19
42	399	521	.3962	098	18
43	427	550	7.3800	094	17
44	.13456	580	.3639	091	16
45	485	609	.3479	087	15
46	514	639	.3319	083	14
47	543	669	.3160	079	13
48	572	698	.3002	075	12
49	600	.13728	.2844	071	11
50	629	758	.2687	.99067	10
51	658	787	7.2531	063	9
52	.13687	817	.2375	059	8
53	716	846	.2220	055	7
54	744	876	.2066	051	6
55	773	906	.1912	047	5
56	802	935	.1759	043	4
57	831	965	.1607	039	3
58	860	.13995	.1455	035	2
59	889	.14024	.1304	031	1
60	.13917	.14054	7.1154	.99027	0
	cos	cot	tan	sin	′

82° READ UP

8° READ DOWN 9°

′	sin	tan	cot	cos	
0	.13917	.14054	7.1154	.99027	60
1	946	084	.1004	023	59
2	.13975	113	.0855	019	58
3	.14004	143	.0706	015	57
4	033	173	.0558	011	56
5	061	202	.0410	006	55
6	090	232	.0264	.99002	54
7	119	262	7.0117	.98998	53
8	148	.14291	6.9972	994	52
9	177	321	.9827	990	51
10	205	351	.9682	986	50
11	.14234	381	.9538	982	49
12	263	410	.9395	978	48
13	292	440	.9252	973	47
14	320	470	.9110	969	46
15	349	499	.8969	965	45
16	378	.14529	.8828	961	44
17	407	559	6.8687	.98957	43
18	436	588	.8548	953	42
19	464	618	.8408	948	41
20	.14493	648	.8269	944	40
21	522	678	.8131	940	39
22	551	707	.7994	936	38
23	580	737	.7856	931	37
24	608	.14767	.7720	927	36
25	637	796	6.7584	.98923	35
26	666	826	.7448	919	34
27	695	856	.7313	914	33
28	.14723	886	.7179	910	32
29	752	915	.7045	906	31
30	781	945	.6912	902	30
31	810	.14975	.6779	897	29
32	838	.15005	.6646	893	28
33	867	034	.8514	[illegible]	27
34	896	064	6.6383	.98884	26
35	925	094	.6252	[illegible]	25
36	954	124	.8122	[illegible]	24
37	.14982	153	.5992	871	23
38	.15011	183	.5863	[illegible]	22
39	040	213	.5734	[illegible]	21
40	069	243	.5606	[illegible]	20
41	097	.15272	.5478	854	19
42	126	302	.5350	[illegible]	18
43	155	332	6.5223	.98845	17
44	184	362	.5097	841	16
45	.15212	391	.4971	[illegible]	15
46	241	421	.4846	832	14
47	270	451	.4721	[illegible]	13
48	299	481	.4596	823	12
49	327	511	.4472	818	11
50	356	.15540	.4348	814	10
51	385	570	.4225	[illegible]	9
52	.15414	600	6.4103	.98805	8
53	442	630	.3980	[illegible]	7
54	471	[illegible]	.3859	[illegible]	6
55	500	689	.3737	791	5
56	529	719	.3617	[illegible]	4
57	557	749	.3496	[illegible]	3
58	586	779	.3376	[illegible]	2
59	615	809	.3257	[illegible]	1
60	.15643	.15838	6.3138	.9875[illegible]	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.15643	.15838	6.3138	.98769	60
1	672	868	.3019	764	59
2	701	898	.2901	760	58
3	730	928	.2783	755	57
4	758	958	.2656	751	56
5	787	.15988	.2549	746	55
6	.15816	.16017	.2432	741	54
7	845	047	.2316	737	53
8	873	077	.2200	732	52
9	902	107	6.2085	728	51
10	931	137	.1970	.98723	50
11	959	167	.1856	718	49
12	.15988	196	.1742	714	48
13	.16017	226	.1628	709	47
14	046	.16256	.1515	704	46
15	074	286	.1402	700	45
16	103	316	.1290	695	44
17	132	346	.1178	690	43
18	160	376	6.1066	686	42
19	189	405	.0955	681	41
20	218	435	.0844	.98676	40
21	246	465	.0734	671	39
22	.16275	.16495	.0624	667	38
23	304	525	.0514	662	37
24	333	555	.0405	657	36
25	361	585	.0296	652	35
26	390	615	.0188	648	34
27	419	645	6.0080	643	33
28	447	674	5.9972	638	32
29	476	704	.9865	633	31
30	505	.16734	.9758	.98629	30
31	.16533	764	.9651	624	29
32	562	794	.9545	619	28
33	591	824	.9439	614	27
34	620	854	.9333	609	26
35	648	884	.9228	604	25
36	677	914	5.9124	600	24
37	706	944	.9019	595	23
38	734	.16974	.8915	590	22
39	.16763	.17004	.8811	585	21
40	792	033	.8708	.98580	20
41	820	063	.8605	575	19
42	849	093	.8502	570	18
43	878	123	.8400	565	17
44	906	153	5.8298	561	16
45	935	183	.8197	556	15
46	964	.17213	.8095	551	14
47	.16992	243	.7994	546	13
48	.17021	273	.7894	541	12
49	050	303	.7794	536	11
50	078	333	.7694	.98531	10
51	107	363	.7594	526	9
52	136	393	5.7495	521	8
53	164	.17423	.7396	516	7
54	.17193	453	.7297	511	6
55	222	483	.7199	506	5
56	250	513	.7101	501	4
57	279	543	.7004	496	3
58	308	573	.6906	491	2
59	336	603	.6809	486	1
60	.17365	.17633	5.6713	.98481	0
	cos	cot	tan	sin	′

81° READ UP 80°

10° READ DOWN 11°

′	sin	tan	cot	cos	
0	.17365	.17633	5.6713	.98481	60
1	393	663	.6617	476	59
2	422	693	.6521	471	58
3	451	723	.6425	466	57
4	479	753	.6329	461	56
5	508	783	.6234	455	55
6	537	.17813	.6140	450	54
7	.17565	843	.6045	445	53
8	594	873	.5951	440	52
9	623	903	.5857	435	51
10	651	933	5.5764	.98430	50
11	680	963	.5671	425	49
12	708	.17993	.5578	420	48
13	737	.18023	.5485	414	47
14	766	053	.5393	409	46
15	.17794	083	.5301	404	45
16	823	113	5209	399	44
17	852	143	5118	394	43
18	880	173	.5026	389	42
19	909	203	.4936	383	41
20	937	233	5.4845	.98378	40
21	966	.18263	.4755	373	39
22	.17995	293	.4665	368	38
23	.18023	323	.4575	362	37
24	052	353	.4486	357	36
25	081	384	.4397	352	35
26	109	414	.4308	347	34
27	138	444	.4219	341	33
28	166	474	.4131	336	32
29	195	.18504	.4043	331	31
30	224	534	5.3955	.98325	30
31	252	564	.3868	320	29
32	.18281	594	.3781	315	28
33	309	624	.3694	310	27
34	338	654	.3607	304	26
35	367	684	.3521	299	25
36	395	714	.3435	294	24
37	424	.18745	.3349	288	23
38	452	775	.3263	283	22
39	481	805	.3178	277	21
40	.18509	835	5.3093	.98272	20
41	538	865	.3008	267	19
42	567	895	.2924	261	18
43	595	925	.2839	256	17
44	624	955	.2755	250	16
45	652	.18986	.2672	245	15
46	681	.19016	.2588	240	14
47	710	046	.2505	234	13
48	738	076	.2422	229	12
49	.18767	106	.2339	223	11
50	795	136	5.2257	.98218	10
51	824	166	.2174	212	9
52	852	197	.2092	207	8
53	881	.19227	.2011	201	7
54	910	257	.1929	196	6
55	938	287	.1848	190	5
56	967	317	.1767	185	4
57	.18995	347	.1686	179	3
58	.19024	378	.1606	174	2
59	052	408	.1526	168	1
60	.19081	.19438	5.1446	.98163	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.19081	.19438	5.1446	.98163	60
1	109	468	.1366	157	59
2	138	498	.1286	152	58
3	167	529	.1207	146	57
4	195	559	.1128	140	56
5	224	589	.1049	135	55
6	252	619	.0970	129	54
7	281	649	.0892	124	53
8	.19309	680	.0814	118	52
9	338	.19710	5.0736	.98112	51
10	366	740	.0658	107	50
11	395	770	.0581	101	49
12	423	801	.0504	096	48
13	452	831	.0427	090	47
14	481	861	.0350	084	46
15	509	891	.0273	079	45
16	.19538	921	.0197	073	44
17	566	952	.0121	067	43
18	595	.19982	5.0045	061	42
19	623	.20012	4.9969	.98056	41
20	652	042	.9894	050	40
21	680	073	.9819	044	39
22	709	103	.9744	039	38
23	737	133	.9669	033	37
24	.19766	164	.9594	027	36
25	794	194	.9520	021	35
26	823	.20224	.9446	016	34
27	851	254	4.9372	010	33
28	880	285	.9298	.98004	32
29	908	315	.9225	.97998	31
30	937	345	.9152	992	30
31	965	376	.9078	987	29
32	.19994	406	.9006	981	28
33	.20022	436	.8933	975	27
34	051	.20466	.8860	969	26
35	079	497	4.8788	963	25
36	108	527	.8716	958	24
37	136	557	.8644	.97952	23
38	165	588	.8573	946	22
39	193	618	.8501	940	21
40	222	648	.8430	934	20
41	250	679	.8359	928	19
42	.20279	709	.8288	922	18
43	307	.20739	.8218	916	17
44	336	770	4.8147	.97910	16
45	364	800	.8077	905	15
46	393	830	.8007	899	14
47	421	861	.7937	893	13
48	450	891	.7867	887	12
49	478	921	.7798	881	11
50	507	952	.7729	875	10
51	.20535	.20982	.7659	869	9
52	563	.21013	4.7591	.97863	8
53	592	043	.7522	857	7
54	620	073	.7453	851	6
55	649	104	.7385	845	5
56	677	134	.7317	839	4
57	706	164	.7249	833	3
58	734	195	.7181	827	2
59	763	225	.7114	821	1
60	.20791	.21256	4.7046	.97815	0
	cos	cot	tan	sin	′

79° READ UP 78°

12° READ DOWN 13°

′	sin	tan	cot	cos	
0	.20791	.21256	4.7046	.97815	60
1	820	286	4.6979	809	59
2	848	316	912	803	58
3	877	347	845	797	57
4	905	377	779	791	56
5	933	408	712	784	55
6	962	438	646	778	54
7	.20990	469	580	772	53
8	.21019	.21499	4.6514	766	52
9	047	529	448	760	51
10	076	560	382	.97754	50
11	104	590	317	748	49
12	132	621	252	742	48
13	161	651	187	735	47
14	189	682	122	729	46
15	218	712	4.6057	723	45
16	.21246	.21743	4.5993	717	44
17	275	773	928	711	43
18	303	804	864	705	42
19	331	834	800	698	41
20	360	864	736	.97692	40
21	388	895	673	686	39
22	417	925	609	680	38
23	445	956	4.5546	673	37
24	474	.21986	483	667	36
25	.21502	.22017	420	661	35
26	530	047	357	655	34
27	559	078	294	648	33
28	587	108	232	642	32
29	616	139	169	636	31
30	644	169	107	.97630	30
31	672	200	4.5045	623	29
32	701	231	4.4983	617	28
33	729	.22261	922	611	27
34	.21758	292	860	604	26
35	786	322	799	598	25
36	814	353	737	592	24
37	843	383	676	585	23
38	871	414	615	579	22
39	899	444	555	573	21
40	928	475	4.4494	.97566	20
41	956	.22505	434	560	19
42	.21985	536	373	553	18
43	.22013	567	313	547	17
44	041	597	253	541	16
45	070	628	194	534	15
46	098	658	134	528	14
47	126	689	075	521	13
48	155	719	4.4015	515	12
49	183	.22750	4.3956	508	11
50	212	781	897	.97502	10
51	240	811	838	496	9
52	.22268	842	779	489	8
53	297	872	721	483	7
54	325	903	662	476	6
55	353	934	604	470	5
56	382	964	546	463	4
57	410	.22995	488	457	3
58	438	.23026	430	450	2
59	467	056	372	444	1
60	.22495	.23087	4.3315	.97437	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.22495	.23087	4.3315	.97437	60
1	523	117	257	430	59
2	552	148	200	424	58
3	580	179	143	417	57
4	608	209	086	411	56
5	637	240	4.3029	404	55
6	665	271	4.2972	398	54
7	693	.23301	916	391	53
8	.22722	332	859	384	52
9	750	363	803	378	51
10	778	393	747	.97371	50
11	807	424	691	365	49
12	835	455	635	358	48
13	863	485	580	351	47
14	892	516	4.2524	345	46
15	920	.23547	468	338	45
16	948	578	413	331	44
17	.22977	608	358	325	43
18	.23005	639	303	318	42
19	033	670	248	311	41
20	062	700	193	.97304	40
21	090	731	139	298	39
22	118	.23762	084	291	38
23	146	793	4.2030	284	37
24	175	823	4.1976	278	36
25	203	854	922	271	35
26	231	885	868	264	34
27	.23260	916	814	257	33
28	288	946	760	251	32
29	316	.23977	706	244	31
30	345	.24008	653	.97237	30
31	373	039	600	230	29
32	401	069	547	223	28
33	429	100	4.1493	217	27
34	458	131	441	210	26
35	.23486	162	388	203	25
36	514	193	335	196	24
37	542	223	282	189	23
38	571	.24254	230	182	22
39	599	285	178	176	21
40	627	316	126	.97169	20
41	656	347	074	162	19
42	684	377	4.1022	155	18
43	712	408	4.0970	148	17
44	.23740	439	918	141	16
45	769	.24470	867	134	15
46	797	501	816	127	14
47	825	532	764	120	13
48	853	562	713	113	12
49	882	593	662	106	11
50	910	624	611	.97100	10
51	938	655	560	093	9
52	966	.24686	4.0509	086	8
53	.23995	717	459	079	7
54	.24023	747	408	072	6
55	051	778	358	065	5
56	079	809	308	058	4
57	108	840	257	051	3
58	136	871	207	044	2
59	164	902	158	037	1
60	.24192	.24933	4.0108	.97030	0
	cos	cot	tan	sin	′

77° READ UP 76°

14° READ DOWN 15°

14°

′	sin	tan	cot	cos	
0	.24192	.24933	4.0108	.97030	60
1	220	964	058	023	59
2	249	.24995	4.0009	015	58
3	277	.25026	3.9959	008	57
4	305	056	910	.97001	56
5	333	087	861	.96994	55
6	362	118	812	987	54
7	390	149	763	980	53
8	418	180	714	973	52
9	.24446	211	665	966	51
10	474	242	617	959	50
11	503	.25273	568	952	49
12	531	304	3.9520	945	48
13	559	335	471	.96937	47
14	587	366	423	930	46
15	615	397	375	923	45
16	644	428	327	916	44
17	672	459	279	909	43
18	700	.25490	232	902	42
19	.24728	521	184	894	41
20	756	552	136	887	40
21	784	583	089	.96880	39
22	813	614	3.9042	873	38
23	841	645	3.8995	866	37
24	869	676	947	858	36
25	897	707	900	851	35
26	925	.25738	854	844	34
27	954	769	807	837	33
28	.24982	800	760	829	32
29	.25010	831	714	.96822	31
30	038	862	3.8667	815	30
31	066	893	621	807	29
32	094	924	575	800	28
33	122	955	528	793	27
34	151	.25986	482	786	26
35	179	.26017	436	778	25
36	207	048	391	771	24
37	.25235	079	3.8345	.96764	23
38	263	110	299	756	22
39	291	141	254	749	21
40	320	172	208	742	20
41	348	203	163	734	19
42	376	235	118	727	18
43	404	.26266	073	719	17
44	.25432	297	3.8028	712	16
45	460	328	3.7983	.96705	15
46	488	359	938	697	14
47	516	390	893	690	13
48	545	421	848	682	12
49	573	452	804	675	11
50	601	483	760	667	10
51	629	515	715	660	9
52	.25657	.26546	3.7671	653	8
53	685	577	627	.96645	7
54	713	608	583	638	6
55	741	639	539	630	5
56	769	670	495	623	4
57	798	701	451	615	3
58	826	733	408	608	2
59	854	764	364	600	1
60	.25882	.26795	3.7321	.96593	0
	cos	cot	tan	sin	′

15°

′	sin	tan	cot	cos	
0	.25882	.26795	3.7321	.96593	60
1	910	826	277	585	59
2	938	857	234	578	58
3	966	888	191	570	57
4	.25994	920	148	562	56
5	.26022	951	105	555	55
6	050	.26982	062	547	54
7	079	.27013	3.7019	540	53
8	107	044	3.6976	532	52
9	135	076	933	524	51
10	163	107	891	.96517	50
11	191	138	848	509	49
12	219	169	806	502	48
13	.26247	201	764	494	47
14	275	232	722	486	46
15	303	.27263	680	479	45
16	331	294	3.6638	471	44
17	359	326	596	463	43
18	387	357	554	456	42
19	415	388	512	448	41
20	443	419	470	.96440	40
21	.26471	451	429	433	39
22	500	.27482	387	425	38
23	528	513	346	417	37
24	556	545	3.6305	410	36
25	584	576	264	402	35
26	612	607	222	394	34
27	640	638	181	386	33
28	668	670	140	379	32
29	696	701	100	371	31
30	.26724	.27732	059	.96363	30
31	752	764	3.6018	355	29
32	780	795	3.5978	347	28
33	808	826	937	340	27
34	836	858	897	332	26
35	864	889	856	324	25
36	892	921	816	316	24
37	920	952	776	308	23
38	948	.27983	736	301	22
39	.26976	.28015	696	293	21
40	.27004	046	3.5656	.96285	20
41	032	077	616	277	19
42	060	109	576	269	18
43	088	140	536	261	17
44	116	172	497	253	16
45	144	203	457	246	15
46	172	.28234	418	238	14
47	200	266	379	230	13
48	228	297	3.5339	222	12
49	256	329	300	214	11
50	.27284	360	261	.96206	10
51	312	391	222	198	9
52	340	423	183	190	8
53	368	.28454	144	182	7
54	396	486	105	174	6
55	424	517	067	166	5
56	452	549	3.5028	158	4
57	480	580	3.4989	150	3
58	508	612	951	142	2
59	536	643	912	134	1
60	.27564	.28675	3.4874	.96126	0
	cos	cot	tan	sin	′

75° READ UP 74°

READ DOWN

16°

′	sin	tan	cot	cos	
0	.27564	.28675	3.4874	.96126	60
1	592	706	836	118	59
2	620	738	798	110	58
3	648	769	760	102	57
4	676	801	722	094	56
5	704	832	684	086	55
6	731	864	646	078	54
7	.27759	895	608	.96070	53
8	787	927	3.4570	062	52
9	815	958	533	054	51
10	843	.28990	495	046	50
11	871	.29021	458	037	49
12	899	053	420	029	48
13	927	084	383	021	47
14	955	116	346	013	46
15	.27983	147	3.4308	.96005	45
16	.28011	179	271	.95997	44
17	039	210	234	989	43
18	067	.29242	197	981	42
19	095	274	160	972	41
20	123	305	124	964	40
21	150	337	087	956	39
22	178	368	050	948	38
23	206	400	3.4014	940	37
24	.28234	432	3.3977	931	36
25	262	.29463	941	.95923	35
26	290	495	904	915	34
27	318	526	868	907	33
28	346	558	832	898	32
29	374	590	796	890	31
30	402	621	759	882	30
31	429	653	723	874	29
32	457	685	687	865	28
33	.28485	.29716	3.3652	.95857	27
34	513	748	616	849	26
35	541	780	580	841	25
36	569	811	544	832	24
37	597	843	509	824	23
38	625	875	473	816	22
39	652	906	435	807	21
40	680	938	402	799	20
41	708	.29970	367	791	19
42	.28736	.30001	3.3332	.95782	18
43	764	033	297	774	17
44	792	065	261	766	16
45	820	097	226	757	15
46	847	128	191	749	14
47	875	160	156	740	13
48	903	192	122	732	12
49	931	224	087	724	11
50	959	255	052	715	10
51	.28987	.30287	3.3017	.95707	9
52	.29015	319	3.2983	698	8
53	042	351	948	690	7
54	070	382	914	681	6
55	098	414	879	673	5
56	126	446	845	664	4
57	154	478	811	656	3
58	182	509	777	647	2
59	209	541	743	639	1
60	.29237	.30573	3.2709	.95630	0
	cos	cot	tan	sin	′

73°

17°

′	sin	tan	cot	cos	
0	.29237	.30573	3.2709	.95630	60
1	265	605	675	622	59
2	293	637	641	613	58
3	321	669	607	605	57
4	348	700	573	596	56
5	376	732	539	588	55
6	404	764	506	579	54
7	432	.30796	3.2472	571	53
8	460	828	438	562	52
9	.29487	860	405	554	51
10	515	891	371	.95545	50
11	543	923	338	536	49
12	571	955	305	528	48
13	599	.30987	272	519	47
14	626	.31019	3.2238	511	46
15	654	051	205	502	45
16	682	083	172	493	44
17	710	115	139	485	43
18	.29737	147	106	476	42
19	765	178	073	467	41
20	793	210	041	.95459	40
21	821	242	3.2008	450	39
22	849	.31274	3.1975	441	38
23	876	306	943	433	37
24	904	338	910	424	36
25	932	370	878	415	35
26	960	402	845	407	34
27	.29987	434	813	398	33
28	.30015	466	780	399	32
29	043	.31498	3.1748	380	31
30	071	530	716	.95372	30
31	098	562	684	363	29
32	126	594	652	354	28
33	154	626	620	345	27
34	182	658	588	337	26
35	209	690	556	328	25
36	237	722	524	319	24
37	.30265	.31754	3.1492	310	23
38	292	786	460	301	22
39	320	818	429	293	21
40	348	850	397	.95284	20
41	376	882	366	275	19
42	403	914	334	266	18
43	431	946	303	257	17
44	459	.31978	3.1271	248	16
45	.30486	.32010	240	240	15
46	514	042	209	231	14
47	542	074	178	222	13
48	570	106	146	213	12
49	597	139	115	204	11
50	625	171	084	.95195	10
51	653	203	053	186	9
52	680	235	3.1022	177	8
53	.30708	.32267	3.0991	168	7
54	736	299	961	159	6
55	763	331	930	150	5
56	791	363	899	142	4
57	819	396	868	133	3
58	816	428	838	124	2
59	874	460	807	115	1
60	.30902	.32492	3.0777	.95106	0
	cos	cot	tan	sin	′

READ UP

72°

18° READ DOWN

′	sin	tan	cot	cos	
0	.30902	.32492	3.0777	.95106	60
1	929	524	746	097	59
2	957	556	716	088	58
3	.30985	588	686	079	57
4	.31012	621	655	070	56
5	040	653	625	061	55
6	068	685	595	.95052	54
7	095	717	565	043	53
8	123	.32749	535	033	52
9	151	782	3.0505	024	51
10	178	814	475	015	50
11	206	846	445	.95006	49
12	233	878	415	.94997	48
13	261	911	385	988	47
14	.31289	943	356	979	46
15	316	.32975	326	970	45
16	344	.33007	296	961	44
17	372	040	3.0267	952	43
18	399	072	237	943	42
19	427	104	208	933	41
20	454	136	178	.94924	40
21	482	169	149	915	39
22	.31510	201	120	906	38
23	537	233	090	897	37
24	565	.33266	061	888	36
25	593	298	032	878	35
26	620	330	3.0003	869	34
27	648	363	2.9974	860	33
28	675	395	945	.94851	32
29	703	427	916	842	31
30	730	460	887	832	30
31	.31758	.33492	858	823	29
32	786	524	829	814	28
33	813	557	800	805	27
34	841	589	772	795	26
35	868	621	2.9743	786	25
36	896	654	714	.94777	24
37	923	686	686	768	23
38	951	.33718	657	758	22
39	.31979	751	629	749	21
40	.32006	783	600	740	20
41	034	816	572	730	19
42	061	848	544	721	18
43	089	881	515	712	17
44	116	913	2.9487	.94702	16
45	144	945	459	693	15
46	171	.33978	431	684	14
47	199	.34010	403	674	13
48	227	043	375	665	12
49	254	075	347	656	11
50	.32282	108	319	646	10
51	309	140	2.9291	637	9
52	337	173	263	.94627	8
53	364	205	235	618	7
54	392	.34238	208	609	6
55	419	270	180	599	5
56	447	303	152	590	4
57	474	335	125	580	3
58	502	368	097	571	2
59	529	400	070	561	1
60	.32557	.34433	2.9042	.94552	0
	cos	cot	tan	sin	′

71° READ UP

19° READ DOWN

′	sin	tan	cot	cos	
0	.32557	.34433	2.9042	.94552	60
1	584	465	2.9015	542	59
2	612	498	2.8987	533	58
3	639	530	960	523	57
4	667	563	933	514	56
5	694	596	905	504	55
6	722	628	878	495	54
7	749	661	851	485	53
8	.32777	693	824	476	52
9	804	.34726	797	466	51
10	832	758	770	.94457	50
11	859	791	2.8743	447	49
12	887	824	716	438	48
13	914	856	689	428	47
14	942	889	662	418	46
15	969	922	636	409	45
16	.32997	954	609	399	44
17	.33024	.34987	582	390	43
18	051	.35020	556	380	42
19	079	052	529	370	41
20	106	085	2.8502	.94361	40
21	134	118	476	351	39
22	161	150	449	342	38
23	189	183	423	332	37
24	.33216	216	397	322	36
25	244	.35248	370	313	35
26	271	281	344	303	34
27	298	314	318	293	33
28	326	346	291	284	32
29	353	379	2.8265	274	31
30	381	412	239	.94264	30
31	.33408	.35445	213	254	29
32	436	477	187	245	28
33	463	510	161	235	27
34	490	543	135	225	26
35	518	576	109	215	25
36	545	608	083	206	24
37	573	641	057	196	23
38	.33600	674	032	186	22
39	627	.35707	2.8006	176	21
40	655	740	2.7980	.94167	20
41	682	772	955	157	19
42	710	805	929	147	18
43	737	838	903	137	17
44	764	871	878	127	16
45	.33792	904	852	118	15
46	819	937	2.7827	108	14
47	846	.35969	801	098	13
48	874	.36002	776	.94088	12
49	901	035	751	078	11
50	929	068	725	068	10
51	956	101	700	058	9
52	.33983	134	675	049	8
53	.34011	167	2.7650	039	7
54	038	.36199	625	029	6
55	065	232	600	019	5
56	093	265	575	.94009	4
57	120	298	550	.93999	3
58	147	331	525	989	2
59	175	364	500	979	1
60	.34202	.36397	2.7475	.93969	0
	cos	cot	tan	sin	′

70° READ UP

20° READ DOWN 21°

′	sin	tan	cot	cos	
0	.34202	.36397	2.7475	.93969	60
1	229	430	450	959	59
2	257	463	425	949	58
3	284	496	400	939	57
4	311	529	376	929	56
5	339	562	351	919	55
6	366	595	326	909	54
7	.34393	628	302	899	53
8	421	661	277	889	52
9	448	.36694	2.7253	879	51
10	475	727	228	.93869	50
11	503	760	204	859	49
12	530	793	179	849	48
13	557	826	155	839	47
14	.34584	859	130	829	46
15	612	892	106	819	45
16	639	925	082	809	44
17	666	958	058	799	43
18	694	.36991	034	789	42
19	721	.37024	2.7009	779	41
20	748	057	2.6985	.93769	40
21	.34775	090	961	759	39
22	803	123	937	748	38
23	830	157	913	738	37
24	857	190	889	728	36
25	884	223	865	718	35
26	912	.37256	841	708	34
27	939	289	818	698	33
28	966	322	2.6794	688	32
29	.34993	355	770	677	31
30	.35021	388	746	.93667	30
31	048	422	723	657	29
32	075	455	699	647	28
33	102	.37488	675	637	27
34	130	521	652	626	26
35	157	554	628	616	25
36	184	588	2.6605	606	24
37	.35211	621	581	596	23
38	239	654	558	585	22
39	266	687	534	575	21
40	293	.37720	511	.93565	20
41	320	754	488	555	19
42	347	787	464	544	18
43	375	820	441	534	17
44	.35402	853	2.6418	524	16
45	429	887	395	514	15
46	456	920	371	503	14
47	484	953	348	493	13
48	511	.37986	325	483	12
49	538	.38020	302	472	11
50	565	053	279	[illegible]	10
51	.35592	086	256	452	9
52	619	120	2.6233	441	8
53	647	153	210	431	7
54	674	186	187	420	6
55	701	220	165	410	5
56	728	253	142	400	4
57	755	286	110	389	3
58	782	320	096	379	2
59	810	353	074	368	1
60	.35837	.38386	2.6051	.93358	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.35837	.38386	2.6051	.93358	60
1	864	420	028	348	59
2	891	453	2.6006	337	58
3	918	487	2.5983	327	57
4	945	520	961	316	56
5	.35973	553	938	306	55
6	.36000	587	916	295	54
7	027	620	893	285	53
8	054	654	871	.93274	52
9	081	.38687	848	264	51
10	108	721	826	253	50
11	135	754	2.5804	243	49
12	162	787	782	232	48
13	190	821	759	222	47
14	217	854	737	211	46
15	.36244	888	715	201	45
16	271	921	693	.93190	44
17	298	955	671	180	43
18	325	.38988	2.5649	169	42
19	352	.39022	627	159	41
20	379	055	605	148	40
21	406	089	583	137	39
22	434	122	561	127	38
23	461	156	539	116	37
24	.36488	190	517	106	36
25	515	223	2.5495	.93095	35
26	542	.39257	473	084	34
27	569	290	452	074	33
28	596	324	430	063	32
29	623	357	408	052	31
30	650	391	386	042	30
31	677	425	365	031	29
32	704	458	2.5343	020	28
33	.36731	.39492	322	.93010	27
34	758	526	300	.92999	26
35	785	559	279	988	25
36	812	593	257	978	24
37	839	626	236	967	23
38	867	660	214	956	22
39	894	694	193	945	21
40	921	.39727	2.5172	935	20
41	948	761	150	924	19
42	.36975	795	129	913	18
43	.37002	829	108	.92902	17
44	029	862	086	892	16
45	056	896	065	881	15
46	083	930	044	870	14
47	110	963	023	859	13
48	137	.39997	2.5002	849	12
49	164	.40031	2.4981	838	11
50	191	065	960	827	10
51	218	098	939	816	9
52	.37245	132	918	.92805	8
53	272	166	897	791	7
54	299	200	876	784	6
55	326	234	2.4855	773	5
56	353	267	834	762	4
57	380	301	813	751	3
58	407	335	792	710	2
59	434	369	772	729	1
60	.37461	.40103	2.4751	.92718	0
	cos	cot	tan	sin	′

69° READ UP 68°

22° READ DOWN 23°

′	sin	tan	cot	cos	
0	.37461	.40403	2.4751	.92718	60
1	488	436	730	707	59
2	515	470	709	697	58
3	542	504	689	686	57
4	569	538	668	675	56
5	595	572	648	664	55
6	622	606	627	653	54
7	649	640	606	642	53
8	676	674	586	631	52
9	703	.40707	2.4566	620	51
10	.37730	741	545	.92609	50
11	757	775	525	598	49
12	784	809	504	587	48
13	811	843	484	576	47
14	838	877	464	565	46
15	865	911	443	554	45
16	892	945	423	543	44
17	919	.40979	403	532	43
18	946	.41013	2.4383	521	42
19	973	047	362	510	41
20	.37999	081	342	.92499	40
21	.38026	115	322	488	39
22	053	149	302	477	38
23	080	183	282	466	37
24	107	217	262	455	36
25	134	.41251	242	444	35
26	161	285	222	432	34
27	188	319	2.4202	421	33
28	215	353	182	410	32
29	241	387	162	399	31
30	.38268	421	142	.92388	30
31	295	455	122	377	29
32	322	.41490	102	366	28
33	349	524	083	355	27
34	376	558	063	343	26
35	403	592	043	332	25
36	430	626	023	321	24
37	456	660	2.4004	310	23
38	483	694	2.3984	299	22
39	.38510	.41728	964	287	21
40	537	763	945	.92276	20
41	564	797	925	265	19
42	591	831	906	254	18
43	617	865	886	243	17
44	644	899	867	231	16
45	671	933	2.3847	220	15
46	698	.41968	828	209	14
47	725	.42002	808	198	13
48	.38752	036	789	186	12
49	778	070	770	175	11
50	805	105	750	.92164	10
51	832	139	731	152	9
52	859	173	2.3712	141	8
53	886	207	693	130	7
54	912	.42242	673	119	6
55	939	276	654	107	5
56	966	310	635	096	4
57	.38993	345	616	085	3
58	.39020	379	597	073	2
59	046	413	578	062	1
60	.39073	.42447	2.3559	.92050	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.39073	.42447	2.3559	.92050	60
1	100	482	539	039	59
2	127	516	520	028	58
3	153	551	501	016	57
4	180	585	483	.92005	56
5	207	619	464	.91994	55
6	234	654	445	982	54
7	260	688	2.3426	971	53
8	287	.42722	407	959	52
9	.39314	757	388	948	51
10	341	791	369	936	50
11	367	826	351	925	49
12	394	860	332	914	48
13	421	894	313	.91902	47
14	448	929	2.3294	891	46
15	474	963	276	879	45
16	501	.42998	257	868	44
17	528	.43032	238	856	43
18	.39555	067	220	845	42
19	581	101	201	833	41
20	608	136	183	822	40
21	635	170	2.3164	.91810	39
22	661	205	146	799	38
23	688	239	127	787	37
24	715	274	109	775	36
25	741	308	090	764	35
26	.39768	.43343	072	752	34
27	795	378	053	741	33
28	822	412	035	729	32
29	848	447	2.3017	.91718	31
30	875	481	2.2998	706	30
31	902	516	980	694	29
32	928	550	962	683	28
33	955	585	944	671	27
34	.39982	620	925	660	26
35	.40008	.43654	907	648	25
36	035	689	889	636	24
37	062	724	2.2871	.91625	23
38	088	758	853	613	22
39	115	793	835	601	21
40	141	828	817	590	20
41	168	862	799	578	19
42	195	897	781	566	18
43	.40221	932	763	555	17
44	248	.43966	745	543	16
45	275	.44001	2.2727	.91531	15
46	301	036	709	519	14
47	328	071	691	508	13
48	355	105	673	496	12
49	381	140	655	484	11
50	408	175	637	472	10
51	.40434	210	620	461	9
52	461	.44244	2.2602	.91449	8
53	488	279	584	437	7
54	514	314	566	425	6
55	541	349	549	414	5
56	567	384	531	402	4
57	594	418	513	390	3
58	621	453	496	378	2
59	647	488	478	366	1
60	.40674	.44523	2.2460	.91355	0
	cos	cot	tan	sin	′

67° READ UP 66°

24° READ DOWN

′	sin	tan	cot	cos	
0	.40674	.44523	2.2460	.91355	60
1	700	555	443	343	59
2	727	593	425	331	58
3	753	627	408	319	57
4	780	662	390	307	56
5	806	697	373	295	55
6	.40833	732	355	283	54
7	860	.44767	338	.91272	53
8	886	802	320	260	52
9	913	837	2.2303	248	51
10	939	872	286	236	50
11	966	907	268	224	49
12	.40992	942	251	212	48
13	.41019	.44977	234	200	47
14	045	.45012	216	188	46
15	072	047	199	.91176	45
16	098	082	182	164	44
17	125	117	2.2165	152	43
18	151	152	148	140	42
19	178	187	130	128	41
20	204	222	113	116	40
21	231	.45257	096	104	39
22	.41257	292	079	.91092	38
23	284	327	062	080	37
24	310	362	045	068	36
25	337	397	028	056	35
26	363	432	2.2011	044	34
27	390	467	2.1994	032	33
28	416	.45502	977	020	32
29	443	538	960	.91008	31
30	469	573	943	.90996	30
31	.41496	608	926	984	29
32	522	643	909	972	28
33	549	678	892	960	27
34	575	713	876	948	26
35	602	.45748	2.1859	936	25
36	628	784	842	924	24
37	655	819	825	.90911	23
38	681	854	808	899	22
39	707	889	792	887	21
40	.41734	924	775	875	20
41	760	960	758	863	19
42	787	.45995	742	851	18
43	813	.46030	2.1725	839	17
44	840	065	708	826	16
45	866	101	692	814	15
46	892	136	675	.90802	14
47	919	171	659	790	13
48	945	206	642	778	12
49	972	242	625	766	11
50	.41998	277	609	753	10
51	.42024	312	592	741	9
52	051	.46348	2.1576	729	8
53	077	383	560	.90717	7
54	104	418	543	704	6
55	130	454	527	692	5
56	156	489	510	680	4
57	183	525	494	668	3
58	209	560	478	655	2
59	235	595	461	643	1
60	.42262	.46631	2.1445	.90631	0
	cos	cot	tan	sin	′

65° READ UP

25° READ DOWN

′	sin	tan	cot	cos	
0	.42262	.46631	2.1445	.90631	60
1	288	666	429	618	59
2	315	702	413	606	58
3	341	737	396	594	57
4	367	772	380	582	56
5	394	808	364	569	55
6	420	843	348	557	54
7	.42446	879	332	545	53
8	473	914	315	532	52
9	499	950	2.1299	520	51
10	525	.46985	283	.90507	50
11	552	.47021	267	495	49
12	578	056	251	483	48
13	604	092	235	470	47
14	.42631	128	219	458	46
15	657	163	203	446	45
16	683	199	187	433	44
17	709	234	171	421	43
18	736	270	2.1155	408	42
19	762	305	139	396	41
20	788	.47341	123	.90383	40
21	.42815	377	107	371	39
22	841	412	092	358	38
23	867	448	076	346	37
24	894	483	060	334	36
25	920	519	044	321	35
26	946	555	028	309	34
27	972	590	2.1013	296	33
28	.42999	626	2.0997	284	32
29	.43025	.47662	981	271	31
30	051	698	965	.90259	30
31	077	733	950	246	29
32	104	769	934	233	28
33	130	805	918	221	27
34	156	840	903	208	26
35	182	876	887	196	25
36	209	912	2.0872	183	24
37	.43235	948	856	171	23
38	261	.47984	840	158	22
39	287	.48019	825	146	21
40	313	055	809	.90133	20
41	340	091	794	120	19
42	366	127	778	108	18
43	392	163	763	095	17
44	.43418	198	2.0748	082	16
45	445	234	732	070	15
46	471	.48270	717	057	14
47	497	306	701	045	13
48	523	342	686	032	12
49	549	378	671	019	11
50	575	414	655	.90007	10
51	602	450	640	.89994	9
52	.43628	486	2.0625	981	8
53	654	.48521	609	968	7
54	680	557	594	956	6
55	706	593	579	943	5
56	733	629	564	930	4
57	759	665	549	918	3
58	785	701	533	905	2
59	811	737	518	892	1
60	.43837	.48773	2.0[illegible]	.89879	0
	cos	cot	tan	sin	′

64° READ UP

26° READ DOWN 27°

′	sin	tan	cot	cos	
0	.43837	.48773	2.0503	.89879	60
1	863	809	488	867	59
2	889	845	473	854	58
3	916	881	458	841	57
4	942	917	443	828	56
5	968	953	428	816	55
6	.43994	.48989	413	803	54
7	.44020	.49026	398	790	53
8	046	062	2.0383	777	52
9	072	098	368	764	51
10	098	134	353	.89752	50
11	124	170	338	739	49
12	151	206	323	726	48
13	177	242	308	713	47
14	203	278	293	700	46
15	.44229	.49315	2.0278	687	45
16	255	351	263	674	44
17	281	387	248	662	43
18	307	423	233	649	42
19	333	459	219	636	41
20	359	495	204	.89623	40
21	385	532	189	610	39
22	.44411	568	174	597	38
23	437	604	160	584	37
24	464	.49640	2.0145	571	36
25	490	677	130	558	35
26	516	713	115	545	34
27	542	749	101	532	33
28	568	786	086	519	32
29	.44594	822	072	506	31
30	620	858	057	.89493	30
31	646	894	042	480	29
32	672	931	028	467	28
33	698	.49967	2.0013	454	27
34	724	.50004	1.9999	441	26
35	750	040	984	428	25
36	.44776	076	970	415	24
37	802	113	955	402	23
38	828	149	941	389	22
39	854	185	926	376	21
40	880	222	912	.89363	20
41	906	258	897	350	19
42	932	295	883	337	18
43	958	.50331	1.9868	324	17
44	.44984	368	854	311	16
45	.45010	404	840	298	15
46	036	441	825	285	14
47	062	477	811	272	13
48	088	514	797	259	12
49	114	550	782	245	11
50	140	587	768	.89232	10
51	166	623	754	219	9
52	.45192	.50660	1.9740	206	8
53	218	696	725	193	7
54	243	733	711	180	6
55	269	769	697	167	5
56	295	806	683	153	4
57	321	843	669	140	3
58	347	879	654	127	2
59	373	916	640	114	1
60	.45399	.50953	1.9626	.89101	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.45399	.50953	1.9626	.89101	60
1	425	.50989	612	087	59
2	451	.51026	598	074	58
3	477	063	584	061	57
4	503	099	570	048	56
5	529	136	556	035	55
6	554	173	542	021	54
7	.45580	209	528	.89008	53
8	606	246	1.9514	.88995	52
9	632	283	500	981	51
10	658	319	486	968	50
11	684	.51356	472	955	49
12	710	393	458	942	48
13	736	430	444	928	47
14	762	467	430	915	46
15	.45787	503	1.9416	902	45
16	813	540	402	.88888	44
17	839	577	388	875	43
18	865	614	375	862	42
19	891	651	361	848	41
20	917	.51688	347	835	40
21	942	724	333	822	39
22	968	761	1.9319	808	38
23	.45994	798	306	795	37
24	.46020	835	292	.88782	36
25	046	872	278	768	35
26	072	909	265	755	34
27	097	946	251	741	33
28	123	.51983	237	728	32
29	149	.52020	1.9223	715	31
30	175	057	210	701	30
31	201	094	196	.88688	29
32	226	131	183	674	28
33	.46252	168	169	661	27
34	278	205	155	647	26
35	304	242	142	634	25
36	330	279	128	620	24
37	355	316	1.9115	607	23
38	381	.52353	101	.88593	22
39	407	390	088	580	21
40	433	427	074	566	20
41	458	464	061	553	19
42	.46484	501	047	539	18
43	510	538	034	526	17
44	536	575	020	512	16
45	561	613	1.9007	.88499	15
46	587	650	1.8993	485	14
47	613	.52687	980	472	13
48	639	724	967	458	12
49	664	761	953	445	11
50	690	798	940	431	10
51	.46716	836	927	417	9
52	742	873	913	404	8
53	767	910	1.8900	.88390	7
54	793	947	887	377	6
55	819	.52985	873	363	5
56	844	.53022	860	349	4
57	870	059	847	336	3
58	896	096	834	322	2
59	921	134	820	308	1
60	.46947	.53171	1.8807	.88295	0
	cos	cot	tan	sin	′

63° READ UP 62°

32° READ DOWN 33°

′	sin	tan	cot	cos	
0	.52992	.62487	1.6003	.84805	60
1	.53017	527	1.5993	789	59
2	041	568	983	774	58
3	066	608	972	759	57
4	091	649	962	743	56
5	115	689	952	728	55
6	140	.62730	941	712	54
7	164	770	931	697	53
8	189	811	921	.84681	52
9	.53214	852	911	666	51
10	238	892	1.5900	650	50
11	263	933	890	635	49
12	288	.62973	880	619	48
13	312	.63014	869	601	47
14	337	055	859	588	46
15	361	095	849	.84573	45
16	.53386	136	839	557	44
17	411	177	829	542	43
18	435	217	818	526	42
19	460	258	808	511	41
20	484	299	1.5798	495	40
21	509	.63340	788	480	39
22	534	380	778	464	38
23	558	421	768	.84448	37
24	.53583	462	757	433	36
25	607	503	747	417	35
26	632	544	737	402	34
27	656	584	727	386	33
28	681	625	717	370	32
29	705	.63666	707	355	31
30	[illegible]	707	[illegible]	[illegible]	30
31	754	748	687	324	29
32	.53779	789	677	308	28
33	804	830	667	292	27
34	828	871	657	277	26
35	[illegible]	912	647	[illegible]	25
36	877	953	637	245	24
37	902	.63994	627	230	23
38	926	.64035	617	.84214	22
39	951	076	607	198	21
40	.53975	117	1.5597	182	20
41	.54000	158	587	167	19
42	024	199	577	151	18
43	049	240	567	135	17
44	073	281	557	120	16
45	097	.64322	547	.84104	15
46	122	363	537	088	14
47	146	404	527	072	13
48	171	446	517	057	12
49	195	487	507	041	11
50	54220	528	1.5497	025	10
51	244	569	487	.84009	9
52	269	.64610	477	.83994	8
53	293	652	468	978	7
54	317	693	458	962	6
55	342	734	448	946	5
56	366	775	438	[illegible]	4
57	391	817	428	915	3
58	415	858	418	899	2
59	440	899	408	883	1
60	.54464	.64941	1.6399	.83867	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.54464	.64941	1.5399	.83867	60
1	488	.64982	389	851	59
2	513	.65024	379	835	58
3	537	065	369	819	57
4	561	106	359	804	56
5	586	148	350	788	55
6	610	189	340	772	54
7	.54635	231	330	.83756	53
8	659	272	1.5320	740	52
9	683	.65314	311	724	51
10	708	355	301	708	50
11	732	397	291	692	49
12	756	438	282	676	48
13	781	480	272	660	47
14	805	521	262	645	46
15	.54829	563	253	.83629	45
16	854	604	1.5243	613	44
17	878	.65646	233	597	43
18	902	688	224	581	42
19	927	729	214	565	41
20	951	771	204	549	40
21	975	813	195	533	39
22	.54999	854	185	517	38
23	.55024	896	175	.83501	37
24	048	938	1.5166	485	36
25	072	.65980	156	469	35
26	097	.66021	147	453	34
27	121	063	137	437	33
28	145	105	127	421	32
29	169	147	118	405	31
30	.55194	189	108	389	30
31	218	230	099	.83373	29
32	242	272	1.5089	356	28
33	266	314	080	340	27
34	291	.66356	070	324	26
35	315	398	061	308	25
36	339	440	051	292	24
37	55363	482	042	276	23
38	388	524	032	260	22
39	412	566	023	.83244	21
40	436	608	013	228	20
41	460	.66650	1.5004	212	19
42	484	692	1.4994	195	18
43	509	734	985	179	17
44	533	776	975	163	16
45	.55557	818	966	147	15
46	581	860	957	131	14
47	605	902	947	.83115	13
48	630	944	938	098	12
49	654	.66986	928	082	11
50	678	.67028	919	066	10
51	702	071	1.4910	050	9
52	.55726	113	900	034	8
53	750	155	891	017	7
54	775	197	882	.83001	6
55	799	239	872	.82985	5
56	823	282	863	969	4
57	847	324	854	953	3
58	871	366	844	936	2
59	895	409	835	920	1
60	.55919	.67451	1.4826	.82904	0
	cos	cot	tan	sin	′

57° READ UP 56°

34° READ DOWN 35°

34°

′	sin	tan	cot	cos	
0	.55919	.67451	1.4826	.82904	60
1	943	493	816	887	59
2	968	536	807	871	58
3	.55992	578	798	855	57
4	.56016	620	788	839	56
5	040	663	779	822	55
6	064	.67705	770	806	54
7	088	748	761	790	53
8	112	790	751	773	52
9	136	832	742	.82757	51
10	160	875	1.4733	741	50
11	184	917	724	724	49
12	.56208	.67960	715	708	48
13	232	.68002	705	692	47
14	256	045	696	675	46
15	280	088	687	659	45
16	305	130	678	643	44
17	329	173	669	626	43
18	353	215	659	.82610	42
19	377	258	650	593	41
20	.56401	301	1.4641	577	40
21	425	.68343	632	561	39
22	449	386	623	544	38
23	473	429	614	528	37
24	497	471	605	511	36
25	521	514	596	495	35
26	545	557	586	478	34
27	569	600	577	.82462	33
28	.56593	.68642	568	446	32
29	617	685	559	429	31
30	641	728	1.4550	413	30
31	665	771	541	396	29
32	689	814	532	380	28
33	713	857	523	363	27
34	736	900	514	347	26
35	760	942	505	330	25
36	.56784	.68985	496	.82314	24
37	808	.69028	487	297	23
38	832	071	478	281	22
39	856	114	469	264	21
40	880	157	1.4460	248	20
41	904	200	451	231	19
42	928	243	442	214	18
43	952	286	433	198	17
44	.56976	.69329	424	181	16
45	.57000	372	415	165	15
46	024	416	406	.82148	14
47	047	459	397	132	13
48	071	502	388	115	12
49	095	545	379	098	11
50	119	588	1.4370	082	10
51	143	631	361	065	9
52	167	.69675	352	048	8
53	.57191	718	344	032	7
54	215	761	335	.82015	6
55	238	804	326	.81999	5
56	262	847	317	982	4
57	286	891	308	965	3
58	310	934	299	949	2
59	334	.69977	290	932	1
60	.57358	.70021	1.4281	.81915	0
	cos	cot	tan	sin	′

35°

′	sin	tan	cot	cos	
0	.57358	.70021	1.4281	.81915	60
1	381	064	273	899	59
2	405	107	264	882	58
3	429	151	255	865	57
4	453	194	246	848	56
5	477	238	237	832	55
6	501	281	229	815	54
7	524	.70325	220	.81798	53
8	548	368	1.4211	782	52
9	.57572	412	202	765	51
10	596	455	193	748	50
11	619	499	185	731	49
12	643	542	176	714	48
13	667	586	167	698	47
14	691	629	158	681	46
15	715	.70673	150	.81664	45
16	738	717	1.4141	647	44
17	762	760	132	631	43
18	.57786	804	124	614	42
19	810	848	115	597	41
20	833	891	106	580	40
21	857	935	097	563	39
22	881	.70979	089	546	38
23	904	.71023	080	.81530	37
24	928	066	1.4071	513	36
25	952	110	063	496	35
26	976	154	054	479	34
27	.57999	198	045	462	33
28	.58023	242	037	445	32
29	047	285	028	428	31
30	070	.71329	019	412	30
31	094	373	011	.81395	29
32	118	417	1.4002	378	28
33	141	461	1.3994	361	27
34	165	505	985	344	26
35	189	549	976	327	25
36	.58212	593	968	310	24
37	236	637	959	293	23
38	260	.71681	951	276	22
39	283	725	942	.81259	21
40	307	769	934	242	20
41	330	813	925	225	19
42	354	857	1.3916	208	18
43	378	901	908	191	17
44	.58401	946	899	174	16
45	425	.71990	891	157	15
46	449	.72034	882	140	14
47	472	078	874	.81123	13
48	496	122	865	106	12
49	519	167	857	089	11
50	543	211	848	072	10
51	567	255	1.3840	055	9
52	.58590	299	831	038	8
53	614	.72344	823	021	7
54	637	388	814	.81004	6
55	661	432	806	.80987	5
56	684	477	798	970	4
57	708	521	789	953	3
58	731	565	781	936	2
59	755	610	772	919	1
60	.58779	.72654	1.3764	.80902	0
	cos	cot	tan	sin	′

55° READ UP 54°

36° READ DOWN 37°

′	sin	tan	cot	cos	
0	.58779	.72654	1.3764	.80902	60
1	802	699	755	885	59
2	826	743	747	867	58
3	849	788	739	850	57
4	873	832	730	833	56
5	896	877	722	816	55
6	920	921	713	799	54
7	943	.72966	705	.80782	53
8	967	.73010	697	765	52
9	.58990	055	688	748	51
10	.59014	100	1.3680	730	50
11	037	144	672	713	49
12	061	189	663	696	48
13	084	234	655	679	47
14	108	278	647	662	46
15	131	.73323	638	.80644	45
16	154	368	630	627	44
17	178	413	622	610	43
18	201	457	613	593	42
19	.59225	502	605	576	41
20	248	547	1.3597	558	40
21	272	592	588	541	39
22	295	.73637	580	.80524	38
23	318	681	572	507	37
24	342	726	564	489	36
25	365	771	555	472	35
26	389	816	547	455	34
27	.59412	861	539	438	33
28	436	906	531	420	32
29	459	951	522	403	31
30	482	.73996	1.3514	.80386	30
31	506	.74041	506	368	29
32	529	086	498	351	28
33	552	131	490	334	27
34	576	176	481	316	26
35	.59599	221	473	299	25
36	622	267	465	282	24
37	646	312	457	.80264	23
38	669	.74357	449	247	22
39	693	402	440	230	21
40	716	447	1.3432	212	20
41	739	492	424	195	19
42	763	538	416	178	18
43	.59786	583	408	160	17
44	809	628	400	143	16
45	832	.74674	392	.80125	15
46	856	719	384	108	14
47	879	764	375	091	13
48	902	810	367	073	12
49	926	855	359	056	11
50	949	900	1.3351	038	10
51	972	946	343	021	9
52	.59995	.74991	335	.80003	8
53	.60019	.75037	327	.79986	7
54	042	082	319	968	6
55	065	128	311	951	5
56	089	173	303	934	4
57	112	219	295	916	3
58	135	264	287	899	2
59	158	310	278	881	1
60	.60182	.75355	1.3270	.79864	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.60182	.75355	1.3270	.79864	60
1	205	401	262	846	59
2	228	447	254	829	58
3	251	492	246	811	57
4	274	538	238	793	56
5	298	584	230	776	55
6	321	629	222	758	54
7	344	.75675	214	741	53
8	367	721	206	723	52
9	.60390	767	1.3198	706	51
10	414	812	190	.79688	50
11	437	858	182	671	49
12	460	904	175	653	48
13	483	950	167	635	47
14	506	.75996	159	618	46
15	529	.76042	151	600	45
16	553	088	143	583	44
17	.60576	134	1.3135	565	43
18	599	180	127	547	42
19	622	226	119	530	41
20	645	272	111	.79512	40
21	668	318	103	494	39
22	691	364	095	477	38
23	714	410	087	459	37
24	738	456	079	441	36
25	761	.76502	072	424	35
26	.60784	548	1.3064	406	34
27	807	594	056	388	33
28	830	640	048	371	32
29	853	686	040	353	31
30	876	733	032	.79335	30
31	899	779	024	318	29
32	922	825	017	300	28
33	945	871	009	282	27
34	968	918	1.3001	264	26
35	.60991	.76964	1.2993	247	25
36	.61015	.77010	985	229	24
37	038	057	977	211	23
38	061	103	970	193	22
39	084	149	962	176	21
40	107	196	954	.79158	20
41	130	242	946	140	19
42	153	289	938	122	18
43	176	.77335	1.2931	105	17
44	.61199	382	923	087	16
45	222	428	915	069	15
46	245	475	907	051	14
47	268	521	900	033	13
48	291	568	892	.79016	12
49	314	615	884	.78998	11
50	337	.77661	876	980	10
51	360	708	869	962	9
52	.61383	754	1.2861	944	8
53	406	801	853	926	7
54	429	848	846	908	6
55	451	895	838	891	5
56	474	941	830	873	4
57	497	.77988	822	855	3
58	520	.78035	815	837	2
59	543	082	807	819	1
60	.61566	.78129	1.2799	.78801	0
	cos	cot	tan	sin	′

53° READ UP 52°

38° READ DOWN 39°

′	sin	tan	cot	cos	
0	.61566	.78129	1.2799	.78801	60
1	589	175	792	783	59
2	612	222	784	765	58
3	635	269	776	747	57
4	658	316	769	729	56
5	681	363	761	711	55
6	704	410	753	694	54
7	726	457	746	676	53
8	749	504	738	.78658	52
9	.61772	.78551	731	640	51
10	795	598	1.2723	622	50
11	818	645	715	604	49
12	841	692	708	586	48
13	864	739	700	568	47
14	887	786	693	550	46
15	909	834	685	.78532	45
16	932	881	677	514	44
17	955	928	670	496	43
18	.61978	.78975	662	478	42
19	.62001	.79022	655	460	41
20	024	070	1.2647	442	40
21	046	117	640	424	39
22	069	164	632	.78405	38
23	092	212	624	387	37
24	115	259	617	369	36
25	138	306	609	351	35
26	160	354	602	333	34
27	.62183	401	594	315	33
28	206	449	587	297	32
29	229	.79496	579	.78279	31
30	251	544	1.2572	261	30
31	274	591	564	243	29
32	297	639	557	225	28
33	320	686	549	206	27
34	342	734	542	188	26
35	.62365	781	534	170	25
36	388	829	527	.78152	24
37	411	877	519	134	23
38	433	924	512	116	22
39	456	.79972	504	098	21
40	479	.80020	1.2497	079	20
41	502	067	489	061	19
42	524	115	482	043	18
43	547	163	475	025	17
44	.62570	211	467	.78007	16
45	592	258	460	.77988	15
46	615	306	452	970	14
47	638	354	445	952	13
48	660	402	437	934	12
49	683	450	430	916	11
50	706	.80498	1.2423	897	10
51	728	546	415	879	9
52	.62751	594	408	.77861	8
53	774	642	401	843	7
54	796	690	393	824	6
55	819	738	386	806	5
56	842	786	378	788	4
57	864	834	371	769	3
58	887	882	364	751	2
59	909	930	356	733	1
60	.62932	.80978	1.2349	.77715	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.62932	.80978	1.2349	.77715	60
1	955	.81027	342	696	59
2	.62977	075	334	678	58
3	.63000	123	327	660	57
4	022	171	320	641	56
5	045	220	312	623	55
6	068	268	305	605	54
7	090	316	298	586	53
8	113	364	290	.77568	52
9	135	413	283	550	51
10	.63158	461	1.2276	531	50
11	180	.81510	268	513	49
12	203	558	261	494	48
13	225	606	254	476	47
14	248	655	247	458	46
15	271	703	239	439	45
16	293	752	232	.77421	44
17	.63316	800	225	402	43
18	338	849	218	384	42
19	361	898	210	366	41
20	383	946	1.2203	347	40
21	406	.81995	196	329	39
22	428	.82044	189	310	38
23	451	092	181	.77292	37
24	.63473	141	174	273	36
25	496	190	167	255	35
26	518	238	160	236	34
27	540	287	153	218	33
28	563	336	145	199	32
29	585	385	138	181	31
30	608	434	1.2131	.77162	30
31	.63630	.82483	124	144	29
32	653	531	117	125	28
33	675	580	109	107	27
34	698	629	102	088	26
35	720	678	095	070	25
36	742	727	088	051	24
37	765	776	081	033	23
38	787	825	074	.77014	22
39	.63810	874	066	.76996	21
40	832	923	1.2059	977	20
41	854	.82972	052	959	19
42	877	.83022	045	940	18
43	899	071	038	921	17
44	922	120	031	903	16
45	944	169	024	884	15
46	966	218	017	.76866	14
47	.63989	268	009	847	13
48	.64011	317	1.2002	828	12
49	033	366	1.1995	810	11
50	056	415	988	791	10
51	078	.83465	981	772	9
52	100	514	974	754	8
53	123	564	967	.76735	7
54	.64145	613	960	717	6
55	167	662	953	698	5
56	190	712	946	679	4
57	212	761	939	661	3
58	234	811	932	642	2
59	256	860	925	623	1
60	.64279	.83910	1.1918	.76604	0
	cos	cot	tan	sin	′

51° READ UP 50°

40° READ DOWN

′	sin	tan	cot	cos	
0	.64279	.83910	1.1918	.76604	60
1	301	.83960	910	586	59
2	323	.84009	903	567	58
3	346	059	896	548	57
4	368	108	889	530	56
5	390	158	882	511	55
6	412	208	875	492	54
7	435	258	868	473	53
8	.64457	307	861	.76455	52
9	479	357	854	436	51
10	501	407	1.1847	417	50
11	524	.84457	840	398	49
12	546	507	833	380	48
13	568	556	826	361	47
14	590	606	819	342	46
15	612	656	812	323	45
16	.64635	706	806	.76304	44
17	657	756	799	286	43
18	679	806	792	267	42
19	701	856	785	248	41
20	723	906	1.1778	229	40
21	746	.84956	771	210	39
22	768	.85006	764	192	38
23	790	057	757	173	37
24	.64812	107	750	.76154	36
25	834	157	743	135	35
26	856	207	736	116	34
27	878	257	729	097	33
28	901	308	722	078	32
29	923	358	715	059	31
30	945	408	1.1708	041	30
31	967	.85458	702	022	29
32	.64983	503	695	.76003	28
33	.65011	559	688	.75984	27
34	033	609	681	[illegible]	26
35	055	660	674	[illegible]	25
36	077	710	667	927	24
37	100	761	660	908	23
38	122	811	653	[illegible]	22
39	144	862	647	[illegible]	21
40	166	012	1.1640	851	20
41	188	.85963	633	[illegible]	19
42	.65210	.86014	626	.75813	18
43	232	061	619	794	17
44	254	115	612	[illegible]	16
45	276	166	606	[illegible]	15
46	298	216	599	739	14
47	320	267	592	[illegible]	13
48	342	318	585	[illegible]	12
49	364	368	578	680	11
50	.65386	419	1.1571	661	10
51	408	.86470	565	.75642	9
52	430	521	558	623	8
53	452	572	551	604	7
54	474	623	544	[illegible]	6
55	496	674	538	[illegible]	5
56	518	725	531	547	4
57	540	[illegible]	524	[illegible]	3
58	562	827	517	[illegible]	2
59	584	878	510	[illegible]	1
60	.65606	.86929	1.1504	.75471	0
	cos	cot	tan	sin	

49° READ UP

41° READ DOWN

′	sin	tan	cot	cos	
0	.65606	.86929	1.1504	.75471	60
1	628	.86980	497	452	59
2	650	.87031	490	433	58
3	672	082	483	414	57
4	694	133	477	395	56
5	716	184	470	375	55
6	738	236	463	356	54
7	759	287	456	337	53
8	781	338	450	.75318	52
9	.65803	389	443	299	51
10	825	441	1.1436	280	50
11	847	.87492	430	261	49
12	869	543	423	241	48
13	891	595	416	222	47
14	913	646	410	203	46
15	935	698	403	184	45
16	956	749	396	.75165	44
17	.65978	801	389	146	43
18	.66000	852	383	126	42
19	022	904	376	107	41
20	044	.87955	1.1369	088	40
21	066	.88007	363	069	39
22	088	059	356	050	38
23	109	110	349	030	37
24	131	162	343	.75011	36
25	.66153	214	336	.74992	35
26	175	265	329	973	34
27	197	317	323	953	33
28	218	369	316	934	32
29	240	421	310	915	31
30	262	.88473	1.1303	896	30
31	284	524	296	876	29
32	.66306	576	290	857	28
33	327	628	283	839	27
34	349	680	276	.74818	26
35	371	732	270	799	25
36	393	784	263	780	24
37	414	836	257	760	23
38	436	888	250	741	22
39	.66458	. 040	243	722	21
40	480	.88992	1.1237	703	20
41	501	.89045	230	683	19
42	523	097	224	664	18
43	545	149	217	.74644	17
44	566	201	211	625	16
45	588	253	204	606	15
46	66610	306	197	586	14
47	632	358	191	567	13
48	653	410	184	548	12
49	675	463	178	528	11
50	697	.89515	1.1171	509	10
51	718	567	165	.74489	9
52	740	620	158	470	8
53	.66762	672	152	451	7
54	783	725	145	431	6
55	805	777	139	412	5
56	827	830	132	392	4
57	848	883	126	373	3
58	870	935	119	353	2
59	891	.89988	113	334	1
60	.66913	.90040	1.1106	.74314	0
	cos	cot	tan	sin	′

READ UP 48°

42° READ DOWN 43°

′	sin	tan	cot	cos	
0	.66913	.90040	1.1106	.74314	60
1	935	093	100	295	59
2	956	146	093	276	58
3	978	199	087	256	57
4	.66999	251	080	237	56
5	.67021	304	074	217	55
6	043	357	067	198	54
7	064	410	061	178	53
8	086	463	1.1054	.74159	52
9	107	.90516	048	139	51
10	129	569	041	120	50
11	151	621	035	100	49
12	172	674	028	080	48
13	.67194	727	022	061	47
14	215	781	016	041	46
15	237	834	009	022	45
16	258	887	1.1003	.74002	44
17	280	940	1.0996	.73983	43
18	301	.90993	990	963	42
19	323	.91046	983	944	41
20	.67344	099	977	924	40
21	366	153	971	904	39
22	387	206	964	885	38
23	409	259	958	865	37
24	430	313	951	846	36
25	452	366	1.0945	.73826	35
26	473	419	939	806	34
27	495	473	932	787	33
28	.67516	.91526	926	767	32
29	538	580	919	747	31
30	559	633	913	728	30
31	580	687	907	708	29
32	602	740	900	688	28
33	623	794	894	669	27
34	645	847	1.0888	.73649	26
35	.67666	901	881	629	25
36	688	.91955	875	610	24
37	709	.92008	869	590	23
38	730	062	862	570	22
39	752	116	856	551	21
40	773	170	850	531	20
41	795	224	843	511	19
42	816	277	1.0837	.73491	18
43	.67837	331	831	472	17
44	859	385	824	452	16
45	880	439	818	432	15
46	901	.92493	812	413	14
47	923	547	805	393	13
48	944	601	799	373	12
49	965	655	793	353	11
50	.67987	709	786	333	10
51	.68008	763	1.0780	.73314	9
52	029	817	774	294	8
53	051	872	768	274	7
54	072	926	761	254	6
55	093	.92980	755	234	5
56	115	.93034	749	215	4
57	136	088	742	195	3
58	157	143	736	175	2
59	179	197	730	155	1
60	.68200	.93252	1.0724	.73135	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.68200	.93252	1.0724	.73135	60
1	221	306	717	116	59
2	242	360	711	096	58
3	264	415	705	076	57
4	285	469	699	056	56
5	306	524	692	036	55
6	327	.93578	686	.73016	54
7	349	633	680	.72996	53
8	370	688	674	976	52
9	.68391	742	668	957	51
10	412	797	1.0661	937	50
11	434	852	655	917	49
12	455	906	649	897	48
13	476	.93961	643	877	47
14	497	.94016	637	857	46
15	518	071	630	.72837	45
16	539	125	624	817	44
17	561	180	618	797	43
18	.68582	235	612	777	42
19	603	290	606	757	41
20	624	345	1.0599	737	40
21	645	400	593	717	39
22	666	455	587	697	38
23	688	.94510	581	.72677	37
24	709	565	575	657	36
25	730	620	569	637	35
26	751	676	562	617	34
27	.68772	731	556	597	33
28	793	786	550	577	32
29	814	841	544	557	31
30	835	896	1.0538	537	30
31	857	.94952	532	517	29
32	878	.95007	526	.72497	28
33	899	062	519	477	27
34	920	118	513	457	26
35	941	173	507	437	25
36	962	229	501	417	24
37	.68983	284	495	397	23
38	.69004	340	489	377	22
39	025	395	483	357	21
40	046	451	1.0477	337	20
41	067	.95506	470	.72317	19
42	088	562	464	297	18
43	109	618	458	277	17
44	130	673	452	257	16
45	.69151	729	446	236	15
46	172	785	440	216	14
47	193	841	434	196	13
48	214	897	428	.72176	12
49	235	.95952	422	156	11
50	256	.96008	1.0416	136	10
51	277	064	410	116	9
52	.69298	120	404	095	8
53	319	176	398	075	7
54	340	232	392	055	6
55	361	288	385	035	5
56	382	344	379	.72015	4
57	403	400	373	.71995	3
58	424	457	367	974	2
59	445	513	361	954	1
60	.69466	.96569	1.0355	.71934	0
	cos	cot	tan	sin	′

47° READ UP 46°

44° READ DOWN 44°

′	sin	tan	cot	cos		′	sin	tan	cot	cos	
0	.69466	.96569	1.0355	.71934	60	30	091	270	1.0176	325	30
1	487	625	349	914	59	31	112	327	170	305	29
2	508	681	343	894	58	32	132	384	164	284	28
3	529	738	337	873	57	33	153	441	158	264	27
4	549	794	331	853	56	34	70174	.98499	152	243	26
5	570	850	325	833	55	35	195	556	147	223	25
6	591	907	319	813	54	36	215	613	141	.71203	24
7	.69612	.96963	313	792	53	37	236	671	135	182	23
8	633	.97020	307	772	52	38	257	728	129	162	22
9	654	076	301	.71752	51	39	277	786	123	141	21
10	675	133	1.0295	732	50	40	298	843	1.0117	121	20
11	696	189	289	711	49	41	319	901	111	100	19
12	717	246	283	691	48	42	.70339	.98958	105	080	18
13	737	302	277	671	47	43	360	.99016	099	059	17
14	758	359	271	650	46	44	381	073	094	039	16
15	779	416	265	630	45	45	401	131	088	.71019	15
16	.69800	.97472	259	610	44	46	422	189	082	.70998	14
17	821	529	253	590	43	47	443	247	076	978	13
18	842	586	247	.71569	42	48	463	304	070	957	12
19	862	643	241	549	41	49	484	362	064	937	11
20	883	700	1.0235	529	40	50	505	420	1.0058	916	10
21	904	756	230	508	39	51	.70525	.99178	052	896	9
22	925	813	224	488	38	52	546	536	047	875	8
23	946	870	218	468	37	53	567	594	041	.70855	7
24	966	927	212	447	36	54	587	652	035	834	6
25	.69987	.97984	206	427	35	55	608	710	029	813	5
26	.70008	.98041	200	407	34	56	628	768	023	793	4
27	029	098	194	.71386	33	57	649	826	017	772	3
28	049	155	188	366	32	58	670	884	012	752	2
29	070	213	182	345	31	59	690	.99942	006	731	1
30	091	270	1.0176	325	30	60	.70711	1.0000	1.0000	.70711	0
	cos	cot	tan	sin	′		cos	cot	tan	sin	′

45° READ UP 45°

LOGARITHM TABLES

No.	Log.	No.	Log.	No.	Log.	No.	Log.	No.	Log.
1	0.000000	21	1.322219	41	1.612784	61	1.785330	81	1.908485
2	0.301030	22	1.342423	42	1.623249	62	1.792392	82	1.913814
3	0.477121	23	1.361728	43	1.633468	63	1.799341	83	1.919078
4	0.602060	24	1.380211	44	1.643453	64	1.806180	84	1.924279
5	0.698970	25	1.397940	45	1.653213	65	1.812913	85	1.929419
6	0.778151	26	1.414973	46	1.662758	66	1.819544	86	1.934498
7	0.845098	27	1.431364	47	1.672098	67	1.826075	87	1.939519
8	0.903090	28	1.447158	48	1.681241	68	1.832509	88	1.944483
9	0.954243	29	1.462398	49	1.690196	69	1.838849	89	1.949390
10	1.000000	30	1.477121	50	1.698970	70	1.845098	90	1.954243
11	1.041393	31	1.491362	51	1.707570	71	1.851258	91	1.959041
12	1.079181	32	1.505150	52	1.716003	72	1.857332	92	1.963788
13	1.113943	33	1.518514	53	1.724276	73	1.863323	93	1.968483
14	1.146128	34	1.531479	54	1.732394	74	1.869232	94	1.973128
15	1.176091	35	1.544068	55	1.740363	75	1.875061	95	1.977724
16	1.204120	36	1.556303	56	1.748188	76	1.880814	96	1.982271
17	1.230449	37	1.568202	57	1.755875	77	1.886491	97	1.986772
18	1.255273	38	1.579784	58	1.763428	78	1.892095	98	1.991226
19	1.278754	39	1.591065	59	1.770852	79	1.897627	99	1.995635
20	1.301030	40	1.602060	60	1.778151	80	1.903090	100	2.000000

N.	0	1	2	3	4	5	6	7	8	9
100	000000	000434	000868	001301	001734	002166	002598	003029	003461	003891
1	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174
2	8600	9026	9451	9876	010300	010724	011147	011570	011993	012415
3	012837	013259	013680	014100	4521	4940	5360	5779	6197	6616
4	7033	7451	7868	8284	8700	9116	9532	9947	020361	020775
5	021189	021603	022016	022428	022841	023252	023664	024075	4486	4896
6	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978
7	9384	9789	030195	030600	031004	031408	031812	032216	032619	033021
8	*033424*	*033826*	*4227*	*4628*	*5029*	*5430*	*5830*	*6230*	*6629*	*7028*
9	7426	7825	8223	8620	9017	9414	9811	040207	040602	040998
110	041393	041787	042182	042576	042969	043362	043755	044148	044540	044932
1	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830
2	9218	9606	9993	050380	050766	051153	051538	051924	052309	052694
3	053078	053463	053846	4230	4613	4996	5378	5760	6142	6524
4	6905	7286	7666	8046	8426	8805	9185	9563	9942	060320
5	000698	061075	061452	061829	062206	062582	062958	063333	063709	4083
6	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815
7	8186	8557	8928	9298	9668	070038	070407	070776	071145	071514
8	071882	072250	072617	072985	073352	3718	4085	4451	4816	5182
9	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819
120	079181	079543	079904	080266	080626	080987	081347	081707	082067	082426
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4	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104
5	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637
6	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169
7	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700
8	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231
9	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761
N.	0	1	2	3	4	5	6	7	8	9

N.	0	1	2	3	4	5	6	7	8	9
820	913814	913867	913920	913973	914026	914079	914132	914184	914237	914290
1	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819
2	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347
3	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875
4	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401
5	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927
6	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453
7	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978
8	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502
9	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026
830	919078	919130	919183	919235	919287	919340	919392	919444	919496	919549
1	9601	9653	9706	9758	9810	9862	9914	9967	920019	920071
2	920123	920170	920228	920280	920332	920384	920436	920489	0541	0593
3	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114
4	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634
5	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154
6	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674
7	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192
8	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710
9	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228
840	924279	924331	924383	924434	924486	924538	924589	924641	924693	924744
1	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261
2	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776
3	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291
4	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805
5	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319
6	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832
7	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345
8	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857
9	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368
850	929419	929470	929521	929572	929623	929674	929725	929776	929827	929879
1	9930	9981	930032	930083	930134	930185	930236	930287	930338	930389
2	930440	930491	0542	0592	0643	0694	0745	0796	0847	0898
3	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407
4	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915
5	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423
6	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930
7	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437
8	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943
9	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448
860	934498	934549	934599	934650	934700	934751	934801	934852	934902	934953
1	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457
2	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960
3	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463
4	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966
5	7016	7066	7117	7167	7217	7267	7317	7367	7418	7468
6	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969
7	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470
8	8520	8570	8620	8670	8720	8770	8820	8870	8920	8970
9	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469
870	939519	969569	939619	939669	939719	939769	939819	939869	939918	939968
1	940018	940068	940118	940168	940218	940267	940317	940367	940417	940467
2	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964
3	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462
4	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958
5	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455
6	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950
7	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445
8	3445	3544	3593	3643	3692	3742	3791	3841	3890	3939
9	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433
N.	0	1	2	3	4	5	6	7	8	9

N.	0	1	2	3	4	5	6	7	8	9
880	944483	944532	944581	944631	944680	944729	944779	944828	944877	944927
1	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419
2	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912
3	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403
4	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894
5	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385
6	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875
7	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364
8	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853
9	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341
890	949390	949439	949488	949536	949585	949634	949683	949731	949780	949829
1	9878	9926	9975	950024	950073	950121	950170	950219	950267	950316
2	950365	950414	950462	0511	0560	0608	0657	0706	0754	0803
3	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289
4	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775
5	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260
6	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744
7	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228
8	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711
9	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194
900	954243	954291	954339	954387	954435	954484	954532	954580	954628	954677
1	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158
2	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640
3	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120
4	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601
5	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080
6	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559
7	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038
8	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516
9	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994
910	959041	959089	959137	959185	959232	959280	959328	959375	959423	959471
1	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947
2	9995	960042	960090	960138	960185	960233	960281	960328	960376	960423
3	960471	0518	0566	0613	0661	0709	0756	0804	0851	0899
4	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374
5	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848
6	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322
7	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795
8	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268
9	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741
920	963788	963835	963882	963929	963977	964024	964071	964118	964165	964212
1	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684
2	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155
3	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625
4	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095
5	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564
6	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033
7	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501
8	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969
9	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436
930	968483	968530	968576	968623	968670	968716	968763	968810	968856	968903
1	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369
2	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835
3	9882	9928	9975	970021	970068	970114	970161	970207	970254	970300
4	970347	970393	970440	0486	0533	0579	0626	0672	0719	0765
5	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229
6	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693
7	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157
8	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619
9	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082
N.	0	1	2	3	4	5	6	7	8	9

N.	0	1	2	3	4	5	6	7	8	9
940	973128	973174	973220	973266	973313	973359	973405	973451	973497	973543
1	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005
2	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466
3	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926
4	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386
5	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845
6	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304
7	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763
8	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220
9	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678
950	977724	977769	977815	977861	977906	977952	977998	978043	978089	978135
1	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591
2	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047
3	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503
4	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958
5	980003	980049	980094	980140	980185	980231	980276	980322	980367	980412
6	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867
7	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320
8	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773
9	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226
960	982271	982316	982362	982407	982452	982497	982543	982588	982633	982678
1	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130
2	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581
3	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032
4	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482
5	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932
6	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382
7	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830
8	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279
9	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727
970	986772	986817	986861	986906	986951	986996	987040	987085	987130	987175
1	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622
2	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068
3	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514
4	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960
5	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405
6	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850
7	9895	9939	9983	990028	990072	990117	990161	990206	990250	990294
8	990339	990383	990428	0472	0516	0561	0605	0650	0694	0738
9	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182
980	991226	991270	991315	991359	991403	991448	991492	991536	991580	991625
1	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067
2	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509
3	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951
4	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392
5	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833
6	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273
7	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713
8	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152
9	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591
990	995635	995679	995723	995767	995811	995854	995898	995942	995986	996030
1	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468
2	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906
3	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343
4	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779
5	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216
6	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652
7	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087
8	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522
9	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957
N.	0	1	2	3	4	5	6	7	8	9

Section IX

ANSWERS TO PROBLEMS

ALGEBRA

(Answers to Problems — Pages 215 to 234 Inclusive)

Positive and Negative Numbers

1. (a) 2, (b) 4; (c) 2; (d) -4, (e) 18, (f) -4; (g) .22; (h) 48.
2. (a) 2; (b) -2; (c) 8, (d) -8, (e) -5; (f) -5.
3. (a) -24; (b) 1680, (c) -72, (d) -560.
4. (a) -4; (b) 4; (c) -9; (d) 7; (e) -35; (f) -24.

Common Fractions

5. (a) 1, (b) 1 5/12; (c) 2 13/60, (d) $\dfrac{3x+2y+z}{12}$
6. (a) 1/5, (b) ½; (c) ⅞; (d) $\dfrac{3x-5y}{15}$
7. (a) 1/3, (b) 5/2, (c) 1; (d) ½.
8. (a) ¾; (b) ⅛; (c) 3/5; (d) 2.
9. (a) 2/3, (b) 1/3, (c) 5/6, (d) a/b.
10. (a) $\dfrac{227}{360}$; (b) ¾; (c) $\dfrac{bx+ay}{ab}$; (d) $\dfrac{bx-ay}{ab}$
11. (a) 17/4, (b) 11/8, (c) 13/3; (d) 8/3.
12. 5/32.
13. 3 3/32.
14. $2\dfrac{53}{56}$ sq. in.
15. 2830 pounds.

Decimal Fractions

16. (a) .75; (b) .50125; (c) 5.56, (d) 3.7.
17. (a) 1.375; (b) 2.6875, (c) .899, (d) 1.35, (e) .125, (f) 2.375.
18. (a) .28125; (b) .7854, (c) 4.0625, (d) 52.095.
19. (a) .7854; (b) 40, (c) 7.25; (d) 22.5.
20. (a) .333; (b) .25; (c) .20; (d) .1667, (e) .14286, (f) .125; (g) .111, (h) .0625; (i) .03125, (j) .01563.
21. 3.927 in.
22. 2.91525 sq in.
23. 680 pounds.
24. 281 pounds.
25. 2800 pounds

Square Root

26. (a) 132; (b) 3361, (c) 2; (d) 4; (e) ¼.
27. 3 in.
28. 35.2 in.
29. 29.15 in
30. 10 in.
31. 22.45 in.

Exponents

32. (a) x^2, 9; (b) x^3, 8; (c) x^5, 32; (d) x, 3.
33. (a) 3.1 x^2; (b) $4x^2$; (c) $125y^3$; (dQ $-12y$.
34. (a) x^7; (b) x^{14}; (c) x^3; (d) $120x^{10}$; (e) $\frac{3}{4}x^{10}$.
35. (a) x^3; (b) x^3; (c) $4x^2y$; (d) $2x^3y$.
36. (a) a; (b) $6y$; (c) $4a^2b^2$.
37. (a) $x^2 + 2xy + y^2$; (b) $1.414x^2$; (c) a^5; (d) x^7; (e) 72; (f) -55.
38. (a) x; (*b*) 2; (c) $1/x^2$; (d) a^2; (e) $x^{3.5}$.
39. (a) 64; b) $7^{4/3}$; (c) 4; (d) $a^{n/5}$.
40. (a) x^3y^4; (b) $81a^3$; (c) $\frac{1}{a^2b^2}$; (d) 1; (e) $\frac{a^2}{b^2}$; (f) $\frac{y^2}{x^3}$; (g) a^5
41. (a) w^3; (*b*) 216.
42. .1841.

Solution of Equations

43. 2
44. 15
45. 8
46. 7
47. 10
48. -2
49. 8
50. 8
51. 1
52. 5
53. 5
54. 1
55. 4
56. 9
57. 11/2
58. 5
59. 5
60. 5
61. 0
62. 11
63. 54.77

Simplification

64. $x = 40$.
65. $x = 1\frac{13}{17}$
66. $x = -1$.
67. (a) $x = 11$.
    (b) $x = 12 + 11b$.
68. (a) $x = -\frac{6}{35}$; (b) $x = -3$; (c) $x = -3/2$
69. (a) $x = 1 - 3/2$; (b) $x = -4(1 + a)$.
70. (a) $x = 48/17$; (b) $x = -10 \pm 1/10\sqrt{161}$
71. $fb = \frac{6M}{bh^2}$
72. $AR = \frac{b}{c}$
73. 30°
74. 104°
75. .140 sq. in.
76. 425 pounds.
77. 252 pounds.
78. $4\frac{1}{3}$
79. 25/9
80. $x = \pm 1$ or ± 2
81. $x = 2, 1$
28. $x = 5$
83. $x = 1$
84. $x = 3$
85. $\frac{-1 \pm \sqrt{11}}{2}$
86. $x = -2, x = -2$
87. $x = \pm 2.499$; $x = 0$
88. $x = \pm 3.162$; $x = 0$
89. $x = 1.405$; $x = .58$
90. $x = 1.355$
91. $x = 2.305$; $x = .715$; $x = -3.025$
92. $x = 1.59$; $x = 4.42$; $x = -2$
93. $x = -1$; $x = 4$
94. $x = 3$; $x = -1$; $x = 1 \pm \sqrt{6}$

Factoring

95. -1
96. $-1, 4$.
97. $-\frac{1}{2} \pm \sqrt{3}$
98. $-\frac{1}{2}$
99. $\frac{2}{3}, 1$
100. $-4/5, -3$
101. $4/5, 3$
102. $1, -\frac{1}{2}, -3$
103. $2x^2 - x - 3$
104. $4n^2 + 4n - 3$
105. $5y^2 + 28y + 15$

106. $8x^3 + 38x^2 + 12x - 90$
107 $9x^2 + 24x + 16$
108 $x^3 + 15x^2 + 75x + 125$
109 $3x + 9$
110. $x^2 - 2x\sqrt{2x+3} + 2x + 3$
111. $2x^2 + 2x\sqrt{x^2+4x+3} + 4x + 3$
112. $3x^2 - 7x + 2$
113. $5x^3 + 2x^2 - 6x + 3$
114. $\dfrac{7x^2+3x+9}{x-2}$
115. $3x + 4$
116. $2a^4 - \dfrac{5a^3}{3} + \dfrac{2a^2}{9} + \dfrac{3a}{12} - \dfrac{1}{8}$
117. $\dfrac{b^4}{18} - \dfrac{5b^3}{36} - \dfrac{b^2}{60} + \dfrac{2b}{15} + \dfrac{1}{25}$
118. 3, 6
119. 5.5 sq in.

Completing the Square

120. $x = 3$ or -13
121. $x = 3$ or 2
122. $x = \dfrac{9}{2}$ or $-\dfrac{11}{2}$
123. $x = \dfrac{1}{6}$ or -1
124 $x = 2$ or $\dfrac{3}{2}$
125. $x = 11$ or $\dfrac{11}{3}$
126 $x = \frac{1}{2} \pm \sqrt{2}$
127. $x = -\dfrac{3}{2} \pm \dfrac{1}{2}\sqrt{\dfrac{43}{3}}$
128. $x = -\dfrac{1}{3} \pm \sqrt{2}$
129. $\dfrac{3}{2}$ or $-\dfrac{4}{3}$
130. 18.96
131. 16.225
132. 17.32
133. 13.86
134. $\dfrac{L}{\rho} = 65$
135. .226
136 $x = -2 \pm \sqrt{3}$
137. $x = 1$ or -3
138. $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{17}$
139. $x = \dfrac{3}{4}$ or $-\dfrac{1}{4}$
140. $x = \frac{1}{2} \pm \dfrac{3}{2}\sqrt{\dfrac{3}{5}}$
141. $x = 4 \pm \sqrt{97}$
142. $x = 0.15 \pm 1.331$
143. $x = 4$ or $-\dfrac{17}{4}$
144. $x = 1.80$ or -0.55
145. $x = 3.48$ or -0.315
146 $x = 9$
147. $x = 3$
148. $x = 10,\ x = -2$
149 $x = 10$
150 $x = -1$
151 $x = \dfrac{-15 \pm \sqrt{45}}{18}$
152. $x = 6$
153 $x = 22,\ x = 10$
154 $x = -2$
155. $x = -9$
156 $x = 3,\ y = 1$
157 $x = 2\frac{1}{2},\ y = \frac{1}{4}$
158 $x = 2,\ y = 4$
159. $x = 6,\ y = 2$
160. $x = -15,\ y = 15$
161. $x = -2,\ y = 5$
$x = 2,\ y = -3$
162 $x = 2,\ y = 1$
163. $x = \dfrac{4}{5},\ y = -\dfrac{3}{5}$
164 $x = 2,\ y = \frac{1}{2}$
165 $x = \pm 3,\ y = 4$
166 $x = 2,\ y = -3$
167. $y = 1,\ x = 2$
168 $x = 3{,}000,\ y = 1{,}000$
169 $A = -100,\ B = 173.21$
170 $x = \pm 4,\ y = \pm 6$
171 $x = 7,\ y = -5$
$x = -5,\ y = 7$
172 $x = \pm 4,\ y = \pm 3$
173. $x = \frac{1}{4},\ y = 1$

174. $x = \frac{1}{2}$, $y = -\frac{1}{3}$, $z = \frac{1}{4}$

175. $x = \frac{5}{2}$, $y = \frac{1}{2}$, $z = \frac{3}{2}$

176. $x = 2$, $y = 3$

177. $R_1 = 12\frac{1}{2}$, $R_2 = 37\frac{1}{2}$

178. $x = \pm 2$, $y = \pm 1$

179. $x = 2$, $y = 3$

180. $x = 2$, $y = 3$

181. $x = \dfrac{2(B-A)}{2\ B-A}$

$x = \dfrac{AB}{2B-A}$

182. $T = 86.6$, $c = -50$

183. $x = 11$, $y = 12$, $z = 13$

184. $x = -\frac{19}{7}$, $y = -\frac{10}{17}$, $z = -1$

185. $x = 1$, $y = -1$, $z = 2$

186. $x = 2$, $y = -4$

187. $x = -1 \pm 2\sqrt{2}$, $y = 8 \pm 4\sqrt{2}$

188. $S = 484$, $T = 11$

189. $x = \pm \dfrac{80}{\sqrt{247}}$, $y = \pm \dfrac{15}{\sqrt{247}}$

190. $x = 4$ or $-\dfrac{12}{23}$

$y = -14$ or $\dfrac{198}{23}$

191. $x = 32.6$, $y = 38.8$

192. $x = \frac{99}{24}$, $y = \pm \frac{9}{8}\sqrt{15}$

193. $x = 1 \pm 2\sqrt{2}$, $y = 4 \pm 2\sqrt{2}$

194. $x = 10$, $y = \frac{1}{9}$

195. $A = 16$, $B = -10$

196. $x = \frac{9}{5}$, $y = \frac{4}{5}$, $z = \frac{17}{5}$

197. $x = \frac{1}{2}$, $y = \frac{1}{3}$, $z = \frac{1}{4}$

198. $x = \frac{37}{16}$, $y = \frac{55}{16}$, $z = -\frac{27}{16}$

199. $x = \frac{1}{2}$, $y = -\frac{1}{3}$, $y = \frac{1}{4}$

200. $x = 3$, $y = 6$, $z = 1$

201. $x = 2$, $y = 8$

202. $A = 5$, $B = 4$

203. $S = 484$, $T = 11$

204. $x = 4$, $y = 7$

$x = -1$, $y = 2$

205. $\tan\theta = \dfrac{V^2}{GR}$

206. $\alpha = 45°$, $\beta = 75°$

207. $\alpha = 51°$, $\beta = 39°$

208. $A = 98\frac{1}{3}$, $B = 36\frac{2}{3}$, $C = 45°$

209. $Dx = 2$, $Dy = 4$, $Dz = 6$

210. $Dx = 8.5513$, $Dy = 4.8651$, $Dz = 7.9411$

211.

Copper	7.266
Manganese	2.595
Magnesium	0.865
Aluminum	162.274
	173.000

212.

Copper	6.96
Manganese	0.87
Magnesium	0.87
Aluminum	165.30
	174.00

213. 2.8 inches

214. 65.52%

215. 31.25%

216. 6.25%

217. 20.60%

218.

Material	*Density*	*S. G.*
Water	62.4	1.00
Spruce	27	.43
Mg	109	1.74
24ST	173	2.77
17ST	174	2.78
Steel	490	7.85

219. 2.83

220. 27.50

221. 11.67

222. .25

223. (a) 76.2*cm*; (b) 64.516*cm*

(c) 22.83 in.; (d) 7.87 in.

224. (a) .196 in.^{2}, (b) 4.5 in.
(c) 19.6 in.^{2}; (d) 7.1 in.
225. (a) 146.67 ft./sec.; (b) 158 m.p.h.
(c) 58.66 ft./sec.; (d) 122.73 m.p.h.
226. (a) 29.92 in.; (b) 20.63 lbs./in.^{2}
(c) 203.6 in., (d) 12.28 lbs./in.^{2}
227. (a) .5235 Radians (b) 1.0 Radians
(c) 42.97° (d) 68.76°
228. (a) 16.09 kilo (b) 15.53 miles
(c) 926.99 kilo (d) 1895.27 miles
229. 450 r.p.m.
230. 1250 r.p.m.
231. 33.3 cu. feet
232. 180
233. 67.4
234. 331.1
235. 880 feet
236. .140625
237. 3231 lbs.
238. (a) 9.32; (b) 31.06; (c) 3106
239. (a) 44.7; (b) 400, (c) 346
240. $L_{100} = 1189$
$L_{150} = 2675$
$L_{200} = 4756$

GEOMETRY

(Answers to Problems — Page 235)

1. 37° 18′ 5″
2. 161° 17′ 1″
3. 9° 19′ 57″
4. 39° 50′ 59″
5. 8° 26′ 16″
6. 167° 24′
7. 146° 55′ 9″
8. 7° 17′ 13″
9. 12° 44′ $49\frac{1}{3}$″
10. 42° 52′ $15\frac{7}{8}$″
11. 1 Radian
12. 62.832
13. 4
15. (a) .244
(b) 1.57
(c) 2.62
(d) 104.6°
(e) 90°
(f) 720°
16. 57° 17′ 45″
17. 2865
18. 90

TRIGONOMETRY

(Answers to Problems — Pages 235 to 240 Inclusive)

Types of Triangles

1. Right and oblique triangles
2. A triangle with one angle equal to 90°.

Elements of Triangles

3. Three sides and three angles.
4. (a) Protractor
(b) Construction with compass and triangle
(c) Equation $R = 180° - (A + B)$; If the remainder is 90° the triangle is a right triangle.

Trigonometric Functions in Right Triangles

5. $\text{Sin}\,\phi = \frac{o}{h}$ $\qquad$ $\text{Cos}\,\phi = \frac{a}{h}$ $\qquad$ $\text{Tan}\,\phi = \frac{o}{a}$

6.

FUNCTION	RECIPROCAL FUNCTION
SIN	COSECANT
COS	SECANT
TAN	COTANGENT

7. $\text{Cosecant}\ \phi = \frac{h}{o}$ $\qquad$ $\text{Secant}\,\phi = \frac{h}{a}$ $\qquad$ $\text{Cotangent}\,\phi = \frac{a}{o}$

Geometric Relations

8. $h^2 = a^2 + o^2$ Refer to Problem 5)
9. (a) $a = 15;\ b = 20;\ c = 25$
   (b) $a = 15;\ b = 6;\ c = 16.1^+$
   (c) $a = 5;\ b = 6;\ c = 7.8^+$
   (d) $a = 5;\ b = 8.6^+;\ c = 10$
10. Hypotenuse = 20

Use of Table of Natural Trigonometric Functions—Interpolation

11. 0.17655
12. 0.86155
15. (a) 36° 52′ 12″
    (b) 53° 7′ 48″
16. (a) $\text{Sin}\,\phi = .6000$
    $\text{Cos}\,\phi = .8000$
    $\text{Tan}\,\phi = .7500$
    (b) $\text{Sin}\,\phi = .8000$
    $\text{Cos}\,\phi = .6000$
    $\text{Tan}\,\phi = 1.3333$
20. 6 ft. 9.067 in.

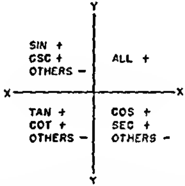
17. 30° (a) Sin .5000
    (b) Cos .8660
    (c) Tan .5774
18. 45° (a) Sin .7071
    (b) Cos .7071
    (c) Tan 1.0000
19. (a) 95.2655
    (b) 129.9075
    (c) 164.5495
13. 1.2282
14. 10° 45′ 30″

Trigonometric Function in Oblique Triangles

21. (Solve graphically)
22. 15° 1st Quadrant
    108° 2nd Quadrant
    210° 3rd Quadrant
    300° 4th Quadrant

23.

	SIN	COS	TAN	COTAN	SEC	COSEC
15°	+.2588	+.9659	+.2679	+3.7321	+1.0353	+3.8637
108°	+.9511	-.3090	-3.0777	-.3249	-3.2361	+1.0515
210°	-.5000	-.8660	+.5774	+1.7321	-1.1547	-2.0000
300°	-.8660	+.5000	-1.7321	-.5774	+2.0000	-1.1547



Geometric Representation of the Trigonometric Functions

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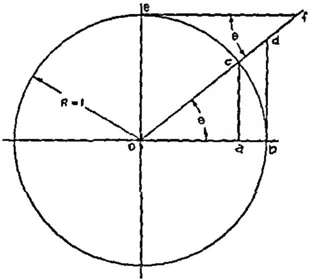

$$\text{Sin} = \frac{ca}{co} = \frac{ca}{1} = ca \qquad \text{Csc} = \frac{ef}{oe} = \frac{ef}{1} = ef$$

$$\text{Cos} = \frac{oa}{oc} = \frac{oa}{1} = oa \qquad \text{Sec} = \frac{od}{ob} = \frac{od}{1} = od$$

$$\text{Tan} = \frac{bd}{ob} = \frac{bd}{1} = bd \qquad \text{CSC} = \frac{of}{oe} = \frac{of}{1} = of$$

Value of the Functions of Obtuse Angles

25. $\sin 100° = +.98481$ $\sin 200° = -.34202$ $\sin 300° = -.86603$
$\cos 100° = -.17365$ $\cos 200° = -.93969$ $\cos 300° = +.50000$
$\tan 100° = -5.6713$ $\tan 200° = +.36397$ $\tan 300° = -1.7321$

Oblique Triangles Solved As Right Triangles

26.

	A	B	C	a	b	c
a)	28°	101°	51°	15.3	32	25.3
b)	21°	153°	6°	57	72.2	16.6
c)	64°	67°	49°	15.5	15.9	13
d)	31° 27'	105° 30'	43° 3'	39	72	51
e)	0°	0°	180°	10	25	35
f)	16°	149° 7'	14° 53'	16.5	36	18
g)	32° 9'	128°	19° 51'	99	146.6	63

Oblique Triangles Solved By Special Formulas

27. Same answers as in Problem 26.

Trigonometric Formulas

30. (a) $\dfrac{\sin\phi + 1}{\cos\phi}$

(b) $\sin^2\phi + \dfrac{1}{\cos^2\phi}$

(c) $\dfrac{\cos^2\phi}{\sin^2\phi} + \dfrac{1}{\sin^2\phi}$

(d) $\sin\phi + \dfrac{1}{\cos\phi}$

(e) $\sin\phi\cos\phi$

31. (a) $\dfrac{1+\sin^2\phi}{1-\sin^2\phi}$

(b) $\dfrac{1}{\sin\phi\,(1-\sin^2\phi)}$

35. (a) -0.2797
(b) 0.0098

36. (a) $\dfrac{56}{65}$

(b) $-\dfrac{33}{65}$

(c) $-\dfrac{36}{77}$

39. (a) $\dfrac{24}{25}$; $\dfrac{7}{25}$; $\dfrac{24}{7}$

(b) $\dfrac{24}{25}$; $\dfrac{7}{25}$; $\dfrac{24}{7}$

(c) $\dfrac{120}{169}$; $-\dfrac{119}{169}$; $\dfrac{120}{119}$

40. (a) $\dfrac{3}{\sqrt{13}}$; $-\dfrac{2}{\sqrt{13}}$; $-\dfrac{3}{2}$

(b) $\dfrac{-2}{\sqrt{13}}$; $\dfrac{3}{\sqrt{13}}$; $-\dfrac{2}{3}$

42. (a) $\sin 7\phi - \sin 3\phi$
(b) $\frac{1}{2}\cos 3\phi + \frac{1}{2}\cos\phi$
(c) $\frac{1}{2}\sin 11\phi - \frac{1}{2}\sin\phi$

43. (a) $-2\cos 45°\sin 5°$
(b) $2\sin 25°\sin 5°$
(c) $2\cos 2\phi\sin\phi$

44. (a) $\cot 30°\tan 10°$
(b) $-\tan\dfrac{3\phi}{2}\tan\dfrac{\phi}{2}$

46. $A = 43°$
$B = 50°$ (to the nearest degree)
$C = 87°$

47. (a) 964.5
(b) 172,695.9
(c) 1,429,431.7

LOGARITHMS

(Answers to Problems—Pages 240 to 244 Inclusive)

Introduction

1. (a) Multiplication
   (b) Division
   (c) Raising to a power
   (d) Extracting a root

Definitions and Principles

2. (a) 3^{10}
   (b) 10^{1}
   (c) 5^{2}
   (d) 10
   (e) 3^{12}
   (f) 3^{0}
   (g) 3^{2}
   (h) $8^{1/3}$

5. (a) 3 = *characteristic*
   (b) 00000 = *mantissa*

6. (a) Mantissa
   (b) Positive

7. (a) $\bar{4}.30103$
   (b) $\bar{3}.30103$
   (c) $\bar{2}.30103$
   (d) $\bar{1}.30103$
   (e) 0.30103
   (f) 1.30103
   (g) 2.30103
   (h) 3.30103
   (j) 4.30103

8. (a) $\bar{3}.66032$
   (b) $\overline{10}.15320$
   (c) −3.17173
   (d) −0.9934

Rules for Characteristics

10. $\bar{4}$, $\bar{3}$, $\bar{2}$, $\bar{1}$, 0, 1, 2, 3; 4

12. (a) 7.66032 − 10
    (b) 10.15320 − 20
    (c) 6.82827 − 10
    (d) 9.00657 − 10

Use of Tables of Logarithms

14. (a) 2.54531
    (b) 1.77525
    (c) 0.99957
    (d) 0.15836
    (e) $\bar{1}.87448$
    (f) $\bar{1}.34635$
    (g) $\bar{3}.80618$
    (h) 3.00945
    (i) 6.30103
    (j) 2.30081

15. (a) 1005.
    (b) 742.0
    (c) 7.95
    (d) 43,700,000
    (e) 0.2400
    (f) 0.016964
    (g) 0.000,000,297,18
    (h) 0.003,894
    (i) 89.73
    (j) 998.1

16. (a) $\bar{2}.47666$
    (b) 0.32504
    (c) 1.518618
    (d) $\bar{1}.602798$
    (e) 0.076388
    (f) 0.890291
    (g) $\bar{3}.82347$
    (h) 6.360704
    (i) 5.9666624
    (j) 0.426302

17. (a) 187.066 (f) 45,445,550.
(b) 12.3257 (g) 94,417.5
(c) 0.052885 (h) 1.035098
(d) 0.0006763666 (i) 2.005727
(e) 0.5102555 (j) 0.2980214

Fundamental Operations Using Logarithms

18. (a) 7,600 (f) 22.9824
(b) 9,240 (g) 2,280,912
(c) 7,736 (h) 0.000,054,08
(d) 86,718 (i) 4.2464
(e) 0.6776 (j) 23,478

19. (a) 1.0575 (f) 0.04762
(b) .5238 (g) 0.00330
(c) 1.2766 (h) 1.0767
(d) 17.0726 (i) 0.6932
(e) 14.5 (j) 15.0

20. (a) 390,625 (f) 3.68715
(b) 46,656 (g) 58.064,400
(c) 86,436 (h) 0.001331
(d) 18.3788 (i) 0.000,000,008
(e) 907,039,232 (j) 792.32

21. (a) 9.1652 (f) 16.4924
(b) 83.318 (g) 4.44796
(c) 10.03326 (h) 3.680
(d) 7.55625 (i) 2.64575
(e) 0.344383 (j) 31.62286

Cologarithms

22. A cologarithm is the logarithm of the reciprocal of a number.

Division or Multiplication of Logarithms

24. No
25. (a) 0.74124 (c) 3.02985 (d) .97346
(b) 2.45065

Solution of Equations Using Logarithms

26. (a) .24444 (b) .0322023 (c) 454.2333
27. (a) .03052 (c) .000,007,23 (d) .0007645
(b) .023298
28. (a) 1,758,520 (b) .779466 (c) .008802
29. (a) 3502.78 (b) 22.1843 (c) 69.7091
30. (a) $a = 61.8$; $b = 102.86$; $B = 59°$
(b) $b = 7948$; $c = 7971.6$; $B = 85° 25'$
(c) $b = 2.221$; $c = 3.118$; $A = 44° 35'$
(d) $a = 13.69$; $c = 21.77$; $A = 38° 58'$
(e) $b = 1.468$; $A = 26° 5'$; $B = 63° 55'$
31. (a) $b = 53.48$; $c = 54.30$; $C = 67° 23'$
(b) $a = 1222$; $c = 1297$; $C = 75° 33'$
(c) $a = 9.368$; $b = 0.1810$; $C = 110° 17'$
(d) $b = 4017$; $c = 2217$; $B = 85° 11'$

32. (a) $b = 675.8$; $B = 100° 2'$; $C = 39° 46'$
    (b) $b = 446.2$; $B = 34°$; $C = 80° 45'$
    $b = 213.2$; $B = 15° 30'$; $C = 99° 15'$
    (c) $B = 90° 0'$; $C = 22° 44'$; $c = 481.7$
    (d) $b = 2218$, $C = 85° 10'$; $B = 33° 23'$
    $b = 1621$; $C^1 = 94° 50'$, $B = 23° 43'$

33. (a) $c = 676$
    (b) $a = 123.3$
    (c) $c = 123.2$, $A = 55° 36'$, $B = 80° 40'$
    (d) $a = 10{,}350$, $B = 59° 18'$, $C = 53° 22'$
    (e) $a = 104.3$, $B = 13° 51'$; $C = 67° 21'$

35. (a) 1 88490 (b) 4 36158 (c) .00260

36. (a) .30101 (b) 2 05341 (c) 62148

37. (a) $\bar{5}$ 3918 (c) 4 63349 (e) 1 28993
    (b) 52039 (d) $\bar{2}$.30886

ANALYTICAL GEOMETRY OF STRAIGHT LINES

(Answers to Problems — Pages 244 to 247 Inclusive)

2. $y = 6$ Straight line parallel to x axis
   $y = 22$ Straight line parallel to x axis.
   $x = 10$ Straight line parallel to y axis
3. Origin
5. A linear equation as all its plotted points fall on a straight line
6. Yes
10. Positive
11. Negative
12. (a) $x = -2$, $y = 4$
    (b) $x = 10$, $y = -20$
    (c) $x = 8$, $y = 2\frac{2}{3}$
    (d) $x = -1$, $y = 1$
13. (a) Slope $+ 2$
    (b) Slope $+ 2$
    (c) Slope $- \frac{4}{3}$
    (d) Slope $+ 1$
14. (a) $\frac{c}{400 - x} = \frac{30}{400}$
    (b) 21.75
15. Parallel lines
16. At right angles
17. Product
18. (a) Parallel
    (b) Parallel
    (c) Perpendicular
    (d) Neither
19. 58 6 minutes
20. $73° 34'$
21. (a) $y = 2x + 4$
    (b) $y = 7x - 28$
    (c) $y = -6\,6x$
    (d) $y = x - 22$
    (e) $y = -2x - 11$
22. (a) $y = 9x + 2$
    (b) $y = -2x + 21$
    (c) $y = 2x + 6$
    (d) $y = -4x + 49$
    (e) $y = x - 2$
23. No
24. Yes

26. (a) Inconsistent
(b) Inconsistent
(c) Independent
(d) Independent
28. (a) 10.08 (b) 3.204 (c) 7.25
29. 23½
30. (a) −3
(b) −1.45

APPENDIX — SLIDE RULE

(Answers to Problems — Pages 247 to 249 Inclusive)

Division

1. .250
2. 4
3. .549
4. 1.822
5. .884
6. 5.333
7. .188
8. .300
9. .899
10. 2.236
11. 2.777
12. 1.436
13. .314
14. 3.185
15. 87.38
16. 16.47
17. .0000420
18. 25
19. .04
20. .00215

Multiplication

21. 3.40
22. 4.03
23. 750
24. 7500
25. 7500
26. .75
27. .075
28. 7.5
29. .00075
30. 77.4
31. 774
32. .1073
33. .00617
34. 2687

35.

	16	17	18	19	20	21	22	23	24
40	640	680	720	760	800	840	880	920	960
41	656	697	738	779	820	861	902	943	984
42	672	714	756	798	840	882	924	966	1008
43	688	731	774	817	860	903	946	989	1032

Combined Multiplication and Division

36. 1.80
37. 51.76
38. 10.11
39. 3101
40. 119.8
41. .0473
42. 1.104
43. 3.246
44. 1.019
45. 961.6

Proportions

46. 50.67
47. 105.4
48. 91.59
49. 13,750
50. 46.62
51. 2.174
52. 5.266
53. 1.464
54. 1.100
55. 74.14
56. .00005016

Square Roots and Squares of Numbers

Find the square root of the following:

57.	.01477	61.	6611	65.	913 2
58.	.4669	62.	.1367	66	81.36
59.	4 669	63.	.05385	67.	19.39
60	2 961	64.	3 048		

Find the squares of the following.

68.	9.672	73	15 92	78	.0000005213
69.	80.28	74	1369	79.	1,040,400
70	10,000	75.	213,400	80	2,490.0
71	3,856	76	00001689	81.	597,500
72.	3 168	77.	.08123	82.	29.48

Square Root of the Sum or the Difference of Two Squares

83.	123 7	85.	101 4	87.	99.50
84.	92 66	86	76.92		

Cube Roots and Cubes of Numbers

88	1 260	95	.02840	102.	125,000
89	2 277	96	4 064	103	599,100
90.	4 135	97.	4 642	104.	24 14
91.	12 62	98	1,368	105	1,000,000
92.	7 0	99	29,790	106.	7 881
93	2 789	100.	103,800	107.	674 5
94.	.2611	101	110,600		

Trigonometric Functions

108.	1 0	114	986286	120.	.974370
109	.707107	115	.002909	121.	.999837
110.	.011926	116	948324	122	.584250
111.	.182808	117	342020	123.	.936672
112.	.019197	118	173648	124.	.039260
113.	.148097	119.	.258819		

Logarithms

125.	0 1584	129.	2 6464	133.	2 960
126.	1.7275	130	118 6	134.	5.972
127.	9.5011 — 10	131	9,510,000,000	135.	23.15
128.	8 8585 — 10	132	15,370,000,000		

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